On Longest Paths and Diameter in Random Apollonian Networks*

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ABSTRACT: We consider the following iterative construction of a random planar triangulation. Start with a triangle embedded in the plane. In each step, choose a bounded face uniformly at random, add a vertex inside that face and join it to the vertices of the face. After n – 3 steps, we obtain a random triangulated plane graph with n vertices, which is called a Random Apollonian Network (RAN). We show that asymptotically almost surely (a.a.s.) a longest path in a RAN has length $o(n)$, refuting a conjecture of Frieze and Tsourakakis. We also show that a RAN always has a cycle (and thus a path) of length $(2n-5)^{\log 2/\log 3}$, and that the expected length of its longest cycles (and paths) is $\Omega\left(n^{0.88}\right)$. Finally, we prove that a.a.s. the diameter of a RAN is asymptotic to $c \log n$, where $c \approx 1.668$ is the solution of an explicit equation. © 2014 Wiley Periodicals, Inc. Random Struct. Alg., 45, 703–725, 2014

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1. INTRODUCTION

Due to the increase of interest in social networks, the Web graph, biological networks etc., in recent years a large amount of research has focused on modelling real world networks (see, e.g., Bonato [1] or Chung and Lu [2]). Despite the outstanding amount of work on models generating graphs with power law degree sequences, a considerably smaller amount of work has focused on generative models for planar graphs. In this paper we study a popular random graph model for generating planar graphs with power law properties, which is defined as follows. Start with a triangle embedded in the plane. In each step, choose a bounded face uniformly at random, add a vertex inside that face and join it to the vertices on the face. We call this operation *subdividing* the face. In this paper, we use the term "face" to refer to a "bounded face," unless specified otherwise. After $n-3$ steps, we have a (random) triangulated plane graph with n vertices and $2n-5$ faces. This is called a *Random Apollonian Network (RAN)* and we study its asymptotic properties, as its number of vertices goes to infinity. The number of edges equals $3n-6$, and hence a RAN is a maximal plane graph.

The term "apollonian network" refers to a deterministic version of this process, formed by subdividing all triangles the same number of times, which was first studied in [3, 4]. Andrade et al. [3] studied power laws in the degree sequences of these networks. Random apollonian networks were defined in Zhou et al. [5] (see Zhang et al. [6] for a generalization to higher dimensions), where it was proved that the diameter of a RAN is asymptotically bounded above by a constant times the logarithm of the number of vertices. It was shown in [5, 7] that RANs exhibit a power law degree distribution. The average distance between two vertices in a typical RAN was shown to be logarithmic by Albenque and Marckert [8]. The degree distribution, k largest degrees and k largest eigenvalues (for fixed k) and the diameter were studied in Frieze and Tsourakakis [9]. We continue this line of research by studying the asymptotic properties of the longest (simple) paths and cycles in RANs and giving sharp estimates for the diameter of a typical RAN.

Before stating our main results, we need a few definitions. In this paper n (respectively, m) always denotes the number of vertices (respectively, faces) of the RAN. All logarithms are in the natural base. We say an event A happens *asymptotically almost surely (a.a.s.)* if $\mathbb{P}[A]$ approaches 1 as n goes to infinity. For two functions $f(n)$ and $g(n)$ we write $f \sim g$ if $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 1$. For a random variable $X = X(n)$ and a function $f(n)$, we say X is *a.a.s. asymptotic to* $f(n)$ (and write *a.a.s.* $X \sim f(n)$) if for every fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[f(n)(1-\varepsilon) \le X \le f(n)(1+\varepsilon)\right] = 1,$$

and we say *a.a.s.* $X = o(f(n))$ if for every fixed $\varepsilon > 0$, $\lim_{n\to\infty}\mathbb{P}\left[X \le \varepsilon f(n)\right] = 1$.

The authors of [9] conjecture in their concluding remarks that a.a.s. a RAN has a path of length $\Omega(n)$. We refute this conjecture by showing the following theorem. Let $\mathcal{L}_m$ be a random variable denoting the number of vertices in a longest path in a RAN with m faces.

Theorem 1.1. *A.a.s. we have* $\mathcal{L}_m = o(m)$.

Recall that a RAN on n vertices has $2n-5$ faces, so Theorem 1.1 implies that a.a.s. a RAN does not have a path of length $\Omega(n)$.

We also prove lower bounds for $\mathcal{L}_m$ deterministically, and its expected value in a RAN. In fact, we will prove lower bounds for $\mathcal{C}_m$ and $\mathbb{E}[\mathcal{C}_m]$, where $\mathcal{C}_m$ denotes the number of vertices in a longest (simple) cycle in a RAN with m faces. This immediately yields lower bounds for $\mathcal{L}_m$ and $\mathbb{E}[\mathcal{L}_m]$ by noting that $\mathcal{L}_m \geq \mathcal{C}_m$ always.

Theorem 1.2. *For every positive integer m, the following statements are true.*

a. $\mathcal{L}_m \geq \mathcal{C}_m \geq m^{\log 2/\log 3} + 2.$
b. $\mathbb{E}[\mathcal{L}_m] \geq \mathbb{E}[\mathcal{C}_m] = \Omega\left(m^{0.88}\right).$

The proofs of Theorems 1.1 and 1.2 are built on two novel graph theoretic observations, valid for all subgraphs of apollonian networks.

We also study the diameter of RANs. In [9] it was shown that the diameter of a RAN is a.a.s. at most $\eta_2 \log n$, where $\eta_2 \approx 7.081$ is the unique solution greater than 1 of $\exp(1/x) = 3e/x$. (Our statement here corrects a minor error in [9], propagated from Broutin and Devroye [10], which stated that η_2 is the unique solution less than 1.) In [8] it was shown that a.a.s. the distance between two randomly chosen vertices of a RAN (which naturally gives a lower bound on the diameter) is asymptotic to $\eta_1 \log n$, where $\eta_1 = 6/11 \approx 0.545$. In this paper, we provide the asymptotic value for the diameter of a typical RAN.

Theorem 1.3. *A.a.s. the diameter of a RAN on n vertices is asymptotic to* $c \log n$*, with* $c = (1-\hat{x}^{-1})/\log h(\hat{x}) \approx 1.668$*, where*

$$h(x) = \frac{12x^3}{1-2x} - \frac{6x^3}{1-x},$$

and $\hat{x} \approx 0.163$ *is the unique solution in the interval (0.1,0.2) to*

$$x(x-1)h'(x) = h(x) \log h(x).$$

The proof of Theorem 1.3 consists of a nontrivial reduction of the problem of estimating the diameter to the problem of estimating the height of a certain skewed random tree, which can be done by applying a result of [10].

We start with some preliminaries in Section 2, and prove Theorems 1.1, 1.2, and 1.3 in Sections 3, 4, and 5, respectively.

2. PRELIMINARIES

The following result is due to Eggenberger and Pólya [11] (see, e.g., Mahmoud [12, Theorem 5.1.2]).

Theorem 2.1. *Start with w white balls and b black balls in an urn. In each step, pick a ball uniformly at random from the urn, look at its colour, and return it to the urn; also add s balls of the same colour to the urn. Let* w_n *and* t_n *be the number of white balls and the total number of balls in the urn after n draws. Then, for any* $\alpha \in [0,1]$ *we have*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{w_n}{t_n} < \alpha\right] = \frac{\Gamma((w+b)/s)}{\Gamma(w/s)\Gamma(b/s)} \int_0^\alpha x^{\frac{w}{s}-1}(1-x)^{\frac{b}{s}-1}\,\mathrm{d}x.$$

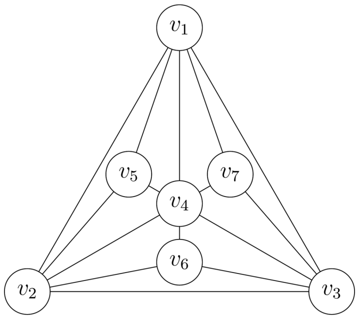


Fig. 1. A triangle in G corresponding to a node of T with nine grandchildren. Vertices $v_1, \ldots, v_7$ are the vertices in the boundaries of the triangles corresponding to these grandchildren.

Note that the right hand side equals $\mathbb{P}\,[\text{Beta}\,(w/s, b/s) < \alpha]$, where Beta (p, q) denotes a beta random variable with parameters p and q. The urn described in Theorem 2.1 is called the *Eggenberger-Pólya* urn.

Let $\triangle$ be a triangle. The *standard 1-subdivision* of $\triangle$ is the set of three triangles obtained from subdividing $\triangle$ once. For $k > 1$, the *standard k-subdivision* of $\triangle$ is the set of triangles obtained from subdividing each triangle in the standard $(k-1)$-subdivision of $\triangle$ exactly once. In Fig. 1, the standard 2-subdivision of a triangle is illustrated.

Consider a triangle $\triangle$ containing more than one face in a RAN, and let $\triangle_1, \triangle_2, \triangle_3$ be the three triangles in its standard 1-subdivision. We can analyze the number of faces inside $\triangle_1$ by modelling the process of building the RAN as an Eggenberger-Pólya urn: after the first subdivision of $\triangle$, each of $\triangle_1$, $\triangle_2$, and $\triangle_3$ contains exactly one face. We start with one white ball corresponding to the only face in $\triangle_1$, and two black balls corresponding to the two faces in $\triangle_2$ and $\triangle_3$. In each subsequent step, we choose a face uniformly at random, and subdivide it. If the face is in $\triangle_1$, then the number of faces in $\triangle_1$ increases by 2, and otherwise the number of faces not in $\triangle_1$ increases by 2. Thus after k subdivisions of $\triangle$, the number of faces in $\triangle_1$ has the same distribution as the number of white balls in an Eggenberger-Pólya urn with $w = 1$, $b = 2$, and $s = 2$, after $k - 1$ draws. This observation leads to the following corollary.

Corollary 2.2. *Let $\triangle$ be a triangle containing m faces in a RAN, and let $Z_1, Z_2, \ldots, Z_9$ be the number of faces inside the nine triangles in the standard 2-subdivision of $\triangle$. Given $\varepsilon > 0$, there exists $m_0 = m_0(\varepsilon)$ such that for $m > m_0$,*

$$\mathbb{P}\,[\min\{Z_1, \ldots, Z_9\}/m < \varepsilon] < 13\sqrt[4]{\varepsilon}.$$

Proof. Let $\overline{\triangle}$ be a triangle containing $\bar{m}$ faces in a RAN, and let W_1, W_2, W_3 be the number of faces inside the three triangles in the standard 1-subdivision of $\overline{\triangle}$. Say that $\overline{\triangle}$ is *balanced* if

$$\min\{W_1, W_2, W_3\}/\bar{m} \geq \sqrt{\varepsilon}.$$

By Theorem 2.1, for a given $1 \leq i \leq 3$ we have

$$\lim_{\bar{m}\to\infty} \mathbb{P}\left[\frac{W_i}{\bar{m}} < \sqrt{\varepsilon}\right] = \int_0^{\sqrt{\varepsilon}} \frac{\Gamma(3/2)}{\Gamma(1)\Gamma(1/2)}\, x^{-1/2}\, \mathrm{d}x = \sqrt{\sqrt{\varepsilon}}.$$

In particular, there exists $\bar{m}_0$ such that

$$\mathbb{P}\left[\frac{W_i}{\bar{m}} < \sqrt{\varepsilon}\right] < \sqrt[4]{1.1\varepsilon}$$

for $\bar{m} > \bar{m}_0$.

Now, take $m_0 = \bar{m}_0/\sqrt{\varepsilon}$, and let $\triangle$ be a triangle containing $m > m_0$ faces in a RAN. The probability that $\triangle$ is balanced is at least $1 - 3\sqrt[4]{1.1\varepsilon}$ by the union bound. If $\triangle$ is balanced, then each of the three triangles in the standard 1-subdivision of $\triangle$ contains more than $m_0\sqrt{\varepsilon} = \bar{m}_0$ faces, so the probability that a certain one of them is not balanced is at most $3\sqrt[4]{1.1\varepsilon}$. Note that if $\triangle$ and these three triangles are balanced, then $\min\{Z_1, \cdots, Z_9\}/m \geq \varepsilon$. Hence by the union bound,

$$\mathbb{P}\left[\min\{Z_1, \ldots, Z_9\}/m < \varepsilon\right] < 12\sqrt[4]{1.1\varepsilon} < 13\sqrt[4]{\varepsilon}. \qquad \blacksquare$$

We include some definitions here. Let G be a RAN. We denote the vertices incident with the unbounded face by ν_1, ν_2, ν_3. All trees we consider are rooted. We define a tree T, called the $\triangle$-*tree* of G, as follows. There is a one to one correspondence between the triangles in G and the nodes of T. For every triangle $\triangle$ in G, we denote its corresponding node in T by $\mathbf{n}^{\triangle}$. To build T, start with a single root node, which corresponds to the triangle $\nu_1\nu_2\nu_3$ of G. Wherever a triangle $\triangle$ is subdivided into triangles $\triangle_1$, $\triangle_2$, and $\triangle_3$, generate three children $\mathbf{n}^{\triangle_1}$, $\mathbf{n}^{\triangle_2}$, and $\mathbf{n}^{\triangle_3}$ for $\mathbf{n}^{\triangle}$, and extend the correspondence in the natural manner. Note that this is a random ternary tree, with each node having either zero or three children, and has $3n - 8$ nodes and $2n - 5$ leaves. We use the term "nodes" for the vertices of T, so that "vertices" refer to the vertices of G. Note that the leaves of T correspond to the faces of G. The *depth* of a node $\mathbf{n}^{\triangle}$ is its distance to the root.

3. UPPER BOUND FOR A LONGEST PATH

In this section we prove Theorem 1.1, stating that a.a.s. all paths in a RAN have length $o(n)$. The set of *grandchildren* of a node is the set of children of its children, so every node in a ternary tree has between zero and nine grandchildren. For a triangle $\triangle$ in G, $I(\triangle)$ denotes the set of vertices of G that are *strictly inside* $\triangle$.

Lemma 3.1. *Let G be a RAN and let T be its $\triangle$-tree. Let $\mathbf{n}^{\triangle}$ be a node of T with nine grandchildren $\mathbf{n}^{\triangle_1}, \mathbf{n}^{\triangle_2}, \ldots, \mathbf{n}^{\triangle_9}$. Then the vertex set of a path in G does not intersect all of the $I(\triangle_i)$'s.*

Proof. There are exactly seven vertices in the boundaries of the triangles corresponding to the grandchildren of $\mathbf{n}^{\triangle}$. Let $v_1, \ldots, v_7$ denote such vertices (see Fig. 1). Let $P = u_1u_2\ldots u_p$ be a path in G. Clearly, when P enters or leaves one of $\triangle_1, \triangle_2, \ldots, \triangle_9$, it must go through a v_i. So P does not contain vertices from more than one triangle between two consecutive occurrences of a v_i. Since P goes through each v_i at most once, the vertices v_i split P up into at most eight sub-paths. Hence P contains vertices from at most eight of the triangles $\triangle_i$. $\blacksquare$

We first sketch a proof of Theorem 1.1. Let G be a RAN on n vertices, and let T be its $\triangle$-tree. The 2-subdivision of the triangle $\nu_1\nu_2\nu_3$ consists of nine triangles, and every path misses the vertices in at least one of them by Lemma 3.1. We can now apply the same argument inductively for the other eight triangles, and repeat. Note that if the distribution of vertices in the nine triangles of every 2-subdivision were always moderately balanced, this argument would immediately prove the theorem (by extending it to $O(\log n)$ depth). Unfortunately, the distribution is biased towards becoming unbalanced: the greater the number of vertices falling in a certain triangle, the higher the probability that the next vertex falls in the same triangle. However, Corollary 2.2 gives an upper bound for the probability that this distribution is very unbalanced. The idea is to use this Corollary iteratively and to use independence of events cleverly to bound the probability of certain "bad" events.

It is easy to see that T is a random ternary recursive tree on $3n-8$ nodes in the sense of Drmota [13]. The following theorem is due to Chauvin and Drmota [14, Theorem 2.3] (we use the wording of [13, Theorem 6.47]).

Theorem 3.2. *Let $\bar{H}_n$ denote the largest number L such that a random n-node ternary recursive tree has precisely 3^L nodes at depth L. Let $\psi \approx 0.152$ be the unique solution in (0,3) to*

$$2\psi \log \left(\frac{3e}{2\psi}\right) = 1.$$

Then we have

$$\mathbb{E}\left[\bar{H}_n\right] \sim \psi \log n,$$

and there exists a constant $\kappa > 0$ such that for every $\varepsilon > 0$,

$$\mathbb{P}\left[|\bar{H}_n - \mathbb{E}\left[\bar{H}_n\right]| > \varepsilon\right] = O(\exp(-\kappa\varepsilon)).$$

Let $D = 0.07 \log n$. Then, the following is obtained immediately.

Corollary 3.3. *A.a.s. there are 3^{2D} nodes at depth $2D$ of T.*

Let $\varepsilon > 0$ be a fixed number such that $3(13\sqrt[4]{4\varepsilon})^{1/5} < 1$, and let $p_F = 1 - 13\sqrt[4]{4\varepsilon}$. Notice that $3(1-p_F)^{1/5} < 1$. We say node $\mathbf{n}^{\triangle}$ is *fair* if at least one of the following holds:

i. the number of faces inside $\triangle$ is less than 3^D, or
ii. $\mathbf{n}^{\triangle}$ has nine grandchildren $\mathbf{n}^{\triangle_1}, \mathbf{n}^{\triangle_2}, \ldots, \mathbf{n}^{\triangle_9}$, and $|I(\triangle_i)| \geq \varepsilon|I(\triangle)|$ for all $1 \leq i \leq 9$.

A triangle $\triangle$ in G is fair if its corresponding node $\mathbf{n}^{\triangle}$ is fair.

Lemma 3.4. *Let $\mathbf{n}^{\triangle}$ be a node in T with nine grandchildren, and let U be a subset of the set of ancestors of $\mathbf{n}^{\triangle}$, not including the parent of $\mathbf{n}^{\triangle}$. The probability that $\mathbf{n}^{\triangle}$ is fair, conditional on all nodes in U being unfair, is at least p_F.*

Proof. Let n be sufficiently large that $3^D > m_0(2\varepsilon)$, where $m_0(2\varepsilon)$ is defined as in Corollary 2.2. Let $\bar{M}$ denote the number of faces inside $\triangle$, and let $\bar{m} \geq 3^D$ be arbitrary. If $\bar{M} < 3^D$, then $\mathbf{n}^{\triangle}$ is fair by definition, so it is enough to prove that

$$\mathbb{P}\left[\mathbf{n}^{\triangle} \text{ is fair} \mid \text{nodes in } U \text{ are unfair}, \bar{M} = \bar{m}\right] \geq p_F.$$

Since U does not contain the parent of $\mathbf{n}^{\triangle}$, conditional on nodes in U being unfair and $\bar{M} = \bar{m}$, the subgraph of G induced by vertices on and inside $\triangle$ is distributed as a RAN with $\bar{m}$ faces.

Let $\triangle_1, \dots, \triangle_9$ be the nine triangles in the standard 2-subdivision of $\triangle$, and let $Z_1, Z_2, \dots, Z_9$ be the number of faces inside them. By Corollary 2.2 and since $\bar{m} > m_0(2\varepsilon)$, with probability at least p_F for all $1 \le i \le 9$,

$$Z_i \ge 2\varepsilon\bar{m},$$

and so

$$I(\triangle_i) = \frac{Z_i - 1}{2} \ge \frac{2\varepsilon\bar{m} - 1}{2} \ge \varepsilon\,\frac{\bar{m} - 1}{2} = \varepsilon I(\triangle),$$

which implies that $\mathbf{n}^{\triangle}$ is fair. ∎

Let $k = (\log\ \log n)/2$. Let $d_0 = 0$ and $d_i = 2^{i-1}k$ for $1 \le i \le k$. Notice that $d_k < D$.

Lemma 3.5. *A.a.s the following is true. Let v be an arbitrary node of T at depth d_i for some $1 \le i \le k$, and let u be the ancestor of v at depth d_{i-1}. Then there is at least one fair node f on the (u, v)-path in T, such that the depth of f is between d_{i-1} and $d_i - 2$, inclusive.*

Proof. Let us say that a node is *bad* if the conclusion of the lemma is false for it. We prove that the probability that a bad node exists is $o(1)$. Let v be a node at depth d_i and u be its ancestor at depth d_{i-1}. Let $x_0 = v, x_1, x_2, \dots, x_r = u$ be the (v, u)-path in T, where $r = d_i - d_{i-1}$. By Lemma 3.4, the probability that none of $x_{2\lfloor r/2\rfloor}, x_{2\lfloor r/2\rfloor-2}, \dots, x_4, x_2$ is fair is at most

$$(1 - p_F)^{\lfloor r/2\rfloor} \le (1 - p_F)^{(d_i - d_{i-1} - 1)/2} \le (1 - p_F)^{d_i/5}.$$

There are at most 3^{d_i} nodes at depth d_i, so by the union bound, the probability that there is at least one bad node v at depth d_i is at most

$$3^{d_i}(1 - p_F)^{d_i/5} = \left[3(1 - p_F)^{1/5}\right]^{d_i} \le \left[3(1 - p_F)^{1/5}\right]^{k} = o(1/k),$$

by the definition of d_i and as $3(1 - p_F)^{1/5} < 1$ and $k \to \infty$. Consequently, the probability that there exists a bad node whose depth lies in $\{d_1, d_2, \dots, d_k\}$ is $o(1)$. ∎

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let G be a RAN with n vertices and m faces, and let T be the $\triangle$-tree of G. The *depth* of a vertex v of G is defined as $\max\{\text{depth}(\triangle) : v \in I(\triangle)\}$, and we define the depth of v_1, v_2, v_3 to be -1. Say a vertex is *deep* if its depth is greater than D, and is *shallow* otherwise. Let n_D denote the number of deep vertices. Note that the number of shallow vertices is at most $(3^{D+1}+5)/2$, which is $o(n)$ by the choice of D, so $n_D = n - o(n)$. For a node $\mathbf{n}^{\triangle}$ of T, let $I_D(\triangle)$ be the set of deep vertices in $I(\triangle)$, and for a subset A of nodes of T, let

$$I_D(A) = \bigcup_{\mathbf{n}^{\triangle} \in A} I_D(\triangle).$$

Claim. If T is full down to depth $2D$ where $D \ge 2$, then any fair node $\mathbf{n}^{\triangle}$ with depth at most D has nine grandchildren $\mathbf{n}^{\triangle_1}, \mathbf{n}^{\triangle_2}, \dots, \mathbf{n}^{\triangle_9}$ such that

$$|I_D(\triangle_i)| \ge \varepsilon|I_D(\triangle)|/2 \qquad i = 1, 2, \dots, 9. \tag{3.1}$$

Proof of Claim. Assume that T is full down to depth $2D$. We first show that for any triangle $\triangle$,

$$|I_D(\triangle)| \geq |I(\triangle)|/2. \tag{3.2}$$

To prove (3.2), let $\triangle$ be a triangle at depth r. If $r > D$, then $I(\triangle)$ contains no shallow vertices, and (3.2) is obviously true. Otherwise, the number of shallow vertices in $I(\triangle)$ equals $1 + 3 + \cdots + 3^{D-r} = (3^{D-r+1} - 1)/2$, whereas the number of vertices in $I(\triangle)$ at depth $D+1$ equals 3^{D-r+1}, where we have used the fact that T is full down to depth at least $D+2$. Thus $I(\triangle)$ contains more deep vertices than shallow vertices, and (3.2) follows.

Now, let $\mathbf{n}^{\triangle}$ be a fair node having depth at most D. Since T is full down to depth $2D$, the number of faces inside $\triangle$ is at least 3^D. So, as $\mathbf{n}^{\triangle}$ is fair, it has nine grandchildren $\mathbf{n}^{\triangle_1}, \mathbf{n}^{\triangle_2}, \ldots, \mathbf{n}^{\triangle_9}$ such that $|I(\triangle_i)| \geq \varepsilon|I(\triangle)|$ for all $1 \leq i \leq 9$. Applying (3.2) gives

$$|I_D(\triangle_i)| \geq |I(\triangle_i)|/2 \geq \varepsilon|I(\triangle)|/2 \geq \varepsilon|I_D(\triangle)|/2 \qquad i = 1, 2, \ldots, 9,$$

as required. ∎

We may condition on two events that happen a.a.s.: the first one is the conclusion of Lemma 3.5, and the second one is that of Corollary 3.3, namely that T is full down to depth $2D$.

To complete the proof of the theorem, for a given path P in G, we will define a sequence $B_0, B_1, \ldots, B_k$ of sets of nodes of T, such that for all $0 \leq i \leq k$ we have

i. $|I_D(B_i)| \geq n_D \left(1 - \left(1 - \frac{\varepsilon}{2}\right)^i\right)$, and
ii. $V(P) \cap I_D(B_i) = \emptyset$.

Before defining the B_i's, let us show that this completes the proof. Notice that (i) gives

$$|I_D(B_k)| \geq n_D - n_D(1 - \varepsilon/2)^k \geq n_D - n_D \exp(-\varepsilon k/2),$$

which is $n - o(n)$ since $n_D = n - o(n)$ and $\varepsilon k = \omega(1)$. Therefore, by (ii),

$$|V(P)| \leq |V(G) \backslash I_D(B_k)| = o(n).$$

So, now we define the sets B_i. Let S_i denote the set of nodes of T at depth d_i. Let $B_0 = \emptyset$ and we define the B_i's inductively, in such a way that $B_i \subseteq S_i$. Fix $1 \leq i \leq k$, and assume that B_{i-1} has already been defined. Let C_i be the set of nodes at depth d_i whose ancestor at depth d_{i-1} is in B_{i-1} (so, in particular, $C_1 = \emptyset$). By the induction hypothesis, $V(P)$ does not intersect $I_D(B_{i-1}) = I_D(C_i)$, and $|I_D(C_i)| = |I_D(B_{i-1})| \geq n_D \left(1 - (1 - \varepsilon/2)^{i-1}\right)$.

Since the conclusion of Lemma 3.5 is true, there exists a set F of fair nodes, with depths between d_{i-1} and $d_i - 2$, such that every $v \in S_i \backslash C_i$ is a descendent of some node in F. Now, for every $x, y \in F$ such that y is a descendent of x, remove y from F. This results in a set $\{u_1, u_2, \ldots, u_s\}$ of fair nodes, with depths between d_{i-1} and $d_i - 2$, such that every $v \in S_i \backslash C_i$ is a descendent of a unique u_j. Recall that $d_k < D$ and so all the u_j's have depths less than D.

Let $w_1, \ldots, w_9$ be the grandchildren of u_1. By Lemma 3.1, $V(P)$ does not intersect all of the $I(w_i)$'s; say it does not intersect $I(w_1)$. Then mark all of the descendants of w_1, and perform a similar procedure for $u_2, \ldots, u_s$. Let M_i be the set of marked nodes in S_i.

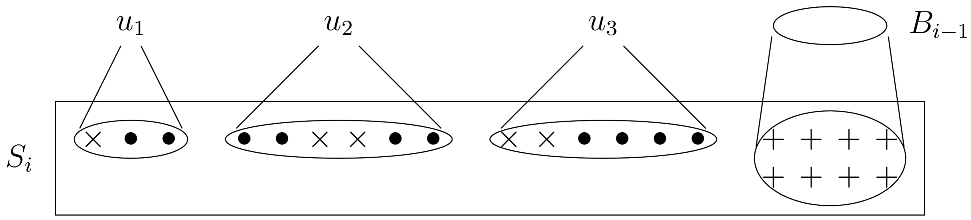


Fig. 2. Illustration for the inductive step in the proof of Theorem 1.1: Vertices in C_i are shown as $+$, vertices in M_i are shown as $\times$, and vertices in $S_i \backslash (C_i \cup M_i)$ are shown as dots.

See Fig. 2. Thus $V(P) \cap I_D(M_i) = \emptyset$. Moreover, since the u_j's are fair and the $I_D(u_j)$'s are disjoint, it follows from the claim that

$$|I_D(M_i)| \geq \sum_{j=1}^{s} \varepsilon |I_D(u_j)|/2 = \varepsilon |I_D(S_i \backslash C_i)|/2 = \varepsilon(n_D - |I_D(B_{i-1})|)/2.$$

Now, let $B_i = C_i \cup M_i$. Then we have

$$\begin{aligned} |I_D(B_i)| &= |I_D(C_i)| + |I_D(M_i)| \\ &\geq |I_D(B_{i-1})| + \frac{\varepsilon}{2}(n_D - |I_D(B_{i-1})|) = |I_D(B_{i-1})|(1 - \frac{\varepsilon}{2}) + \frac{\varepsilon n_D}{2} \\ &\geq n_D\left(1 - \left(1 - \frac{\varepsilon}{2}\right)^{i-1}\right)\left(1 - \frac{\varepsilon}{2}\right) + \frac{\varepsilon n_D}{2} = n_D\left(1 - \left(1 - \frac{\varepsilon}{2}\right)^{i}\right), \end{aligned}$$

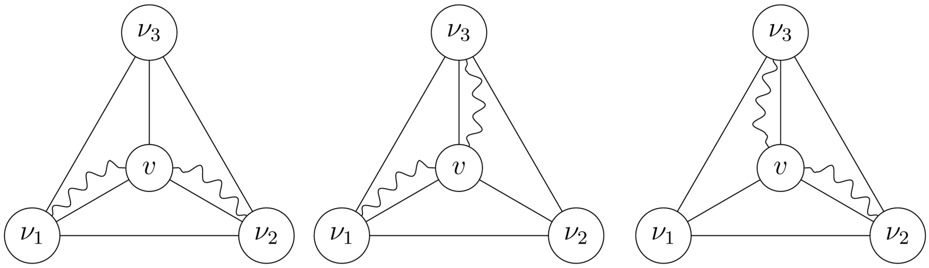
and $V(P)$ does not intersect $I_D(B_i)$. ∎

Remark. *Noting that $n - n_D < n^{1-\delta}$ for some fixed $\delta > 0$ and being more careful in the calculations above shows that indeed we have a.a.s. $\mathcal{L}_m \leq n(\log n)^{-\Omega(1)}$.*

4. LOWER BOUNDS FOR A LONGEST CYCLE (PATH)

In this section we prove Theorem 1.2. We first prove part (a), i.e., we give a deterministic lower bound for the length of a longest cycle in a RAN. Recall that $\mathcal{C}_m$ denotes the number of vertices of a longest cycle in a RAN with m faces. Let G be a RAN with m faces, and let v be the unique vertex that is adjacent to v_1, v_2, and v_3. For $1 \leq i \leq 3$, let $\triangle_i$ be the triangle with vertex set $\{v, v_1, v_2, v_3\} \backslash \{v_i\}$. Define the random variable $\mathcal{L}'_m$ as the largest number L such that for every permutation π on $\{1, 2, 3\}$, there is a path in G of L edges from $v_{\pi(1)}$ to $v_{\pi(2)}$ not containing $v_{\pi(3)}$. Clearly we have $\mathcal{C}_m \geq \mathcal{L}'_m + 2$.

Proof of Theorem 1.2(a). Let $\xi = \log 2/\log 3$. We prove by induction on m that $\mathcal{L}'_m \geq m^{\xi}$. This is obvious for $m = 1$, so assume that $m > 1$. Let m_i denote the number of faces in $\triangle_i$. Then $m_1 + m_2 + m_3 = m$. By symmetry, we may assume that $m_1 \geq m_2 \geq m_3$. For any given $1 \leq i \leq 3$, it is easy to find a path avoiding v_i that connects the other two v_j's by attaching two appropriate paths in $\triangle_1$ and $\triangle_2$ at vertex v. (See Figs. 3a–c.) By the induction hypothesis, these paths can be chosen to have lengths at least ${m_1}^{\xi}$ and ${m_2}^{\xi}$, respectively.



(a) Path avoiding ν_3. (b) Path avoiding ν_2. (c) Path avoiding ν_1

Fig. 3. Paths avoiding $\triangle_3$ and one of the ν_i's.

Hence for every permutation π of $\{1,2,3\}$, there is a path from $\nu_{\pi(1)}$ to $\nu_{\pi(2)}$ avoiding $\nu_{\pi(3)}$ with length at least

$$m_1{}^{\xi} + m_2{}^{\xi}. \tag{4.1}$$

It is easily verified that since $m_1 \geq m_2 \geq m_3$ and $m_1 + m_2 + m_3 = m$, the minimum of (4.1) happens when $m_1 = m_2 = m/3$, thus

$$\mathcal{L}'_m \geq m_1{}^{\xi} + m_2{}^{\xi} \geq 2\left(\frac{m}{3}\right)^{\xi} = m^{\xi},$$

and the proof is complete. ∎

Next, we use the same idea to give a larger lower bound for $\mathbb{E}[\mathcal{C}_m]$. Let the random variable X_i denote the number of faces in $\triangle_i$. Then the X_i's have the same distribution and are not independent. It follows from Theorem 2.1 that as m grows, the distribution of $\frac{X_i}{m}$ converges pointwise to that of Beta$(1/2, 1)$. Moreover, for any fixed $\varepsilon \in [0, 1)$, if we condition on $X_1 = \varepsilon m$, then the subdividing process inside $\triangle_2$ and $\triangle_3$ can be modelled as an Eggenberger-Pólya urn again, and it follows from Theorem 2.1 that the distribution of $\frac{X_2}{(1-\varepsilon)m}$ conditional on $X_1 = \varepsilon m$ converges pointwise to that of Beta$(1/2, 1/2)$. Namely, for any fixed $\varepsilon \in [0, 1)$ and $\delta \in [0, 1]$,

$$\lim_{m\to\infty} \mathbb{P}\left[\frac{X_2}{(1-\varepsilon)m} \leq \delta \middle| X_1 = \varepsilon m\right] = \int_0^{\delta} \frac{\Gamma(1)}{\Gamma(1/2)^2} x^{-1/2}(1-x)^{-1/2}\,\mathrm{d}x. \tag{4.2}$$

We are now ready to prove part (b) of Theorem 1.2.

Proof of Theorem 1.2(b). Let $\zeta = 0.88$. We prove that there exists a constant $\kappa > 0$ such that $\mathbb{E}\left[\mathcal{L}'_m\right] \geq \kappa m^{\zeta}$ holds for all $m \geq 1$. We proceed by induction on m, with the induction base being $m = m_0$, where m_0 is a sufficiently large constant, to be determined later. By choosing κ sufficiently small, we may assume $\mathbb{E}\left[\mathcal{L}'_m\right] \geq \kappa m^{\zeta}$ for all $m \leq m_0$.

For $1 \leq i \leq 3$, let X_i denote the number of faces in $\triangle_i$. Define a permutation σ on $\{1,2,3\}$ such that $X_{\sigma(1)} \geq X_{\sigma(2)} \geq X_{\sigma(3)}$, breaking ties randomly. Then σ is a random permutation determined by the X_i and the random choice in the tie-breaking. By symmetry, for every fixed $\sigma' \in S_3$, $\mathbb{P}\left[\sigma = \sigma'\right] = 1/6$. From the proof of part (a), we know

$$\mathcal{L}'_m \geq \mathcal{L}'_{X_{\sigma(1)}} + \mathcal{L}'_{X_{\sigma(2)}}.$$

Taking the expectation on both sides, we have

$$\mathbb{E}\left[\mathcal{L}'_m\right] \geq \mathbb{E}\left[\mathcal{L}'_{X_{\sigma(1)}} + \mathcal{L}'_{X_{\sigma(2)}}\right] \geq 6\mathbb{E}\left[(\mathcal{L}'_{X_1} + \mathcal{L}'_{X_2})\mathbb{1}_{X_1>X_2>X_3}\right], \tag{4.3}$$

where the second inequality holds by symmetry and as $\mathbb{P}[\sigma = (1,2,3)] = 1/6$. By the induction hypothesis, for every $x_1, x_2 < m$,

$$\mathbb{E}\left[\mathcal{L}'_{X_1}\middle|X_1 = x_1\right] \geq \kappa x_1^{\zeta}, \quad \text{and} \quad \mathbb{E}\left[\mathcal{L}'_{X_2}\middle|X_2 = x_2\right] \geq \kappa x_2^{\zeta}.$$

Hence,

$$\mathbb{E}\left[(\mathcal{L}'_{X_1} + \mathcal{L}'_{X_2})\mathbb{1}_{X_1>X_2>X_3}\right] \geq \kappa\mathbb{E}\left[(X_1^{\zeta} + X_2^{\zeta})\mathbb{1}_{X_1>X_2>X_3}\right]. \tag{4.4}$$

Let $f_1(x)$ and $f_2(x)$ denote the probability density functions of Beta $(1/2, 1)$ and Beta $(1/2, 1/2)$, respectively. Namely,

$$f_1(x) = \frac{\Gamma(3/2)}{\Gamma(1)\Gamma(1/2)}x^{-1/2} \quad \text{and} \quad f_2(x) = \frac{\Gamma(1)}{\Gamma(1/2)^2}x^{-1/2}(1-x)^{-1/2}.$$

Then it follows from Theorem 2.1 that for any fixed $0 \leq t < 1$,

$$\lim_{m\to\infty} \mathbb{P}\left[\frac{X_1}{m} \leq t\right] = \int_0^t f_1(x)\,\mathrm{d}x,$$

and for any fixed $0 \leq s \leq 1$, by (4.2),

$$\lim_{m\to\infty} \mathbb{P}\left[\frac{X_2}{m} \leq (1-t)s\middle|X_1 = tm\right] = \lim_{m\to\infty} \mathbb{P}\left[\frac{X_2}{(1-t)m} \leq s\middle|\frac{X_1}{m} = t\right] = \int_0^s f_2(x)\,\mathrm{d}x.$$

Hence (see Billingsley [15, Theorem 29.1 (i)])

$$\mathbb{E}\left[\left(\left(\frac{X_1}{m}\right)^{\zeta} + \left(\frac{X_2}{m}\right)^{\zeta}\right)\mathbb{1}_{X_1>X_2>X_3}\right] \to \int_{t=1/3}^{1} \int_{s=1/2}^{\min\left\{1,\frac{t}{1-t}\right\}} \left[t^{\zeta} + (s(1-t))^{\zeta}\right] f_1(t)f_2(s)\mathrm{d}s\mathrm{d}t,$$

as $m \to \infty$. By the choice of ζ, we have

$$\int_{t=1/3}^{1} \int_{s=1/2}^{\min\{1,\frac{t}{1-t}\}} \left[t^{\zeta} + (s(1-t))^{\zeta}\right] f_1(t)f_2(s)\,\mathrm{d}s\,\mathrm{d}t > 1/6.$$

Then, by (4.3) and (4.4),

$$\mathbb{E}\left[\mathcal{L}'_m\right] \geq 6\kappa\mathbb{E}\left[(X_1^{\zeta} + X_2^{\zeta})\mathbb{1}_{X_1>X_2>X_3}\right] > \kappa m^{\zeta},$$

if we choose m_0 sufficiently large. ∎

5. DIAMETER

As mentioned in the introduction, prior to this work it had been known that a typical RAN has logarithmic diameter, and asymptotic lower and upper bounds for the diameter had been proved, but the asymptotic value had not been determined. In this section we prove Theorem 1.3, which states that a.a.s. the diameter of a RAN is asymptotic to $c \log n$, where $c \approx 1.668$ is the solution of an explicit equation.

Let G be a RAN with n vertices, and recall that v_1, v_2, and v_3 denote the vertices incident with the unbounded face. For a vertex v of G, let $\tau(v)$ be the minimum graph distance of v to the boundary, i.e.,

$$\tau(v) = \min\{\text{dist}(v, v_1), \text{dist}(v, v_2), \text{dist}(v, v_3)\}.$$

The *radius* of G is defined as the maximum of $\tau(v)$ over all vertices v.

Lemma 5.1. *Let*

$$h(x) = \frac{12x^3}{1-2x} - \frac{6x^3}{1-x},$$

and let $\hat{x}$ be the unique solution in (0.1,0.2) to

$$x(x-1)h'(x) = h(x) \log h(x).$$

Finally, let

$$c = \frac{1-\hat{x}^{-1}}{\log h(\hat{x})} \approx 1.668.$$

Then the radius of G is a.a.s. asymptotic to $c \log n/2$.

We first show that this lemma implies Theorem 1.3.

Proof of Theorem 1.3. Let $\triangle_1$, $\triangle_2$, and $\triangle_3$ be the three triangles in the standard 1-subdivision of the triangle $v_1v_2v_3$, and let n_i be the number of vertices on and inside $\triangle_i$. Let diam (G) denote the diameter of G. Fix arbitrarily small $\varepsilon, \delta > 0$. We show that with probability at least $1 - 2\delta$ we have

$$(1-\varepsilon)c \log n \le \text{diam}(G) \le (1+\varepsilon)c \log n.$$

Here and in the following, we assume n is sufficiently large.

Let M be a positive integer sufficiently large that, for a given $1 \le i \le 3$,

$$\mathbb{P}\left[\frac{n_i}{n} < \frac{1}{M}\right] < \delta/6.$$

Such an M exists by Theorem 2.1 and the discussion after it. Let A denote the event

$$\min\left\{\frac{n_i}{n} : 1 \le i \le 3\right\} \ge \frac{1}{M}.$$

By the union bound, $\mathbb{P}[A] \geq 1 - \delta/2$. We condition on values (n_1, n_2, n_3) such that A happens. Note that we have $\log n_i = \log n - O(1)$ for each i.

For a triangle $\triangle$, $V(\triangle)$ denotes the three vertices of $\triangle$. Note that for $1 \leq i \leq 3$, the subgraph induced by vertices on and inside $\triangle_i$ is distributed as a RAN G_i with n_i vertices. Hence by Lemma 5.1 and the union bound, with probability at least $1 - \delta/2$, the radius of each of G_1, G_2 and G_3 is at least $(1 - \varepsilon)c \log n/2$. Hence, with probability at least $1 - \delta/2$ there exists $u_1 \in V(G_1)$ with distance at least $(1 - \varepsilon)c \log n/2$ to $V(\triangle_1)$, and also there exists $u_2 \in V(G_2)$ with distance at least $(1 - \varepsilon)c \log n/2$ to $V(\triangle_2)$. Since any (u_1, u_2)-path must contain a vertex from $V(\triangle_1)$ and $V(\triangle_2)$, with probability at least $1 - \delta/2$, there exists $u_1, u_2 \in V(G)$ with distance at least $2(1 - \varepsilon)c \log n/2$, which implies

$$\mathbb{P}\left[\operatorname{diam}(G) \geq c(1 - \varepsilon) \log n\right] \geq \mathbb{P}\left[\operatorname{diam}(G) \geq c(1 - \varepsilon) \log n \middle| A\right] \mathbb{P}[A] > 1 - \delta.$$

For the upper bound, let R be the radius of G. Notice that the distance between any vertex and v_1 is at most $R+1$, so $\operatorname{diam}(G) \leq 2R+2$. By Lemma 5.1, with probability at least $1-\delta$ we have $R \leq (1+\varepsilon/2)c \log n/2$. If this event happens, then $\operatorname{diam}(G) \leq (1+\varepsilon)c \log n$. ∎

The rest of this section is devoted to the proof of Lemma 5.1. Let T be the $\triangle$-tree of G, as defined in Section 2. We categorize the triangles in G into three types. Let $\triangle$ be a triangle in G with vertex set $\{x, y, z\}$, and assume that $\tau(x) \leq \tau(y) \leq \tau(z)$. Since z and x are adjacent, we have $\tau(z) \leq \tau(x) + 1$. So, $\triangle$ can be categorized to be of one of the following types:

1. if $\tau(x) = \tau(y) = \tau(z)$, then say $\triangle$ is of type 1.
2. If $\tau(x) = \tau(y) < \tau(y) + 1 = \tau(z)$, then say $\triangle$ is of type 2.
3. If $\tau(x) < \tau(x) + 1 = \tau(y) = \tau(z)$, then say $\triangle$ is of type 3.

The type of a node of T is the same as the type of its corresponding triangle. The root of T corresponds to the triangle $v_1 v_2 v_3$ and the following are easy to observe.

a. The root is of type 1.
b. A node of type 1 has three children of type 2.
c. A node of type 2 has one child of type 2 and two children of type 3.
d. A node of type 3 has two children of type 3 and one child of type 1.

For a triangle $\triangle$, define $\tau(\triangle)$ to be the minimum of $\tau(u)$ over all $u \in V(\triangle)$. Observe that a node of type 1 or 2 has children with the same τ value as itself; whereas for a node of type 3, the τ value of the type-1 child equals the τ value of the parent plus one, and the two other children have the same τ value as the parent. Let $\overline{\triangle}$ and $\triangle$ be two triangles of type 1 such that $\mathbf{n}^{\overline{\triangle}}$ is an ancestor of $\mathbf{n}^{\triangle}$ and there is no node of type 1 in the unique path connecting them. Then, the internal vertices of the path connecting $\mathbf{n}^{\overline{\triangle}}$ and $\mathbf{n}^{\triangle}$ consists of a sequence of type-2 nodes and then a sequence of type-3 nodes, hence we have $\tau(\triangle) = \tau(\overline{\triangle}) + 1$. This determines τ inductively: for every $\mathbf{n}^{\triangle} \in V(T)$, $\tau(\triangle)$ is one less than the number of nodes of type 1 in the path from $\mathbf{n}^{\triangle}$ to the root. We call $\tau(\triangle)$ the *auxiliary depth* of node $\mathbf{n}^{\triangle}$, and define the *auxiliary height* of a tree T, written $\operatorname{ah}(T)$, to be the maximum auxiliary depth of its nodes. Note that the auxiliary height is always less than or equal to the height. Also, for a vertex $v \in V(G)$, if $\triangle$ is the triangle that v subdivides, then $\tau(v) = \tau(\triangle) + 1$. We augment the tree T by adding specification of the type of each node, and we abuse notation and call the augmented tree the $\triangle$-tree of the RAN. Hence, the radius of the RAN is either $\operatorname{ah}(T)$ or $\operatorname{ah}(T) + 1$.

Notice that instead of building T from the RAN G, one can think of the random T as being generated in the following manner: let $n \geq 3$ be a positive integer. Start with a single node as the root of T. So long as the number of nodes is less than $3n - 8$, choose a leaf v independently of previous choices and uniformly at random, and add three leaves as children of v. Once the number of nodes becomes $3n - 8$, add the information about the types using rules (a)–(d), as follows. Let the root have type 1, and determine the types of other nodes in a top-down manner. For a node of type 1, let its children have type 2. For a node of type 2, select one of the children independently and uniformly at random, let that child have type 2, and let the other two children have type 3. Similarly, for a node of type 3, select one of the children independently of previous choices and uniformly at random, let that child have type 1, and let the other two children have type 3. Henceforth, we will forget about G and focus on finding the auxiliary height of a random tree T generated in this manner.

A major difficulty in analyzing the auxiliary height of the tree generated in the aforementioned manner is that the branches of a node are heavily dependent, as the total number of nodes equals $3n - 8$. To remedy this we consider another process which has the desired independence and approximates the original process well enough for our purposes. The process, $\hat{P}$, starts with a single node, the root, which is born at time 0, and is of type 1. From this moment onwards, whenever a node is born (say at time κ), it waits for a random time X, which is distributed exponentially with mean 1, and after time X has passed (namely, at absolute time $\kappa + X$) gives birth to three children, whose types are determined as before (according to the rules (b)–(d), and using randomness whenever there is a choice) and dies. Moreover, the lifetime of the nodes are independent. By the memorylessness of the exponential distribution, if one starts looking at the process at any (deterministic) moment, the next leaf to die is chosen uniformly at random. For a nonnegative (possibly random) t, we denote by $\hat{T}^t$ the random almost surely finite tree obtained by taking a snapshot of this process at time t. Hence, for any deterministic $t \geq 0$, the distribution of $\hat{T}^t$ conditional on $\hat{T}^t$ having exactly $3n - 8$ nodes, is the same as the distribution of T.

Lemma 5.2. *Assume that there exists a constant c such that a.a.s. the auxiliary height of $\hat{T}^t$ is asymptotic to ct as $t \to \infty$. Then the radius of a RAN with n vertices is a.a.s. asymptotic to $c \log n/2$ as $n \to \infty$.*

Proof. Let $\ell_n = 3n - 8$, and let $\varepsilon > 0$ be fixed. For the process $\hat{P}$, we define three stopping times as follows:

a_1 is the deterministic time $(1-\varepsilon) \log(\ell_n)/2$.
A_2 is the random time when the evolving tree has exactly ℓ_n nodes.
a_3 is the deterministic time $(1+\varepsilon) \log(\ell_n)/2$.

Broutin and Devroye [10, Proposition 2] proved that almost surely

$$\log |V(\hat{T}^t)| \sim 2t,$$

which implies the same statement a.a.s. as $t \to \infty$. This means that, as $n \to \infty$, a.a.s.

$$\log |V(\hat{T}^{a_1})| \sim 2a_1 = (1-\varepsilon) \log(\ell_n),$$

and hence $|V(\hat{T}^{a_1})| < \ell_n$, which implies $a_1 < A_2$. Symmetrically, it can be proved that a.a.s. as $n \to \infty$ we have $A_2 < a_3$. It follows that a.a.s. as $n \to \infty$

$$\mathrm{ah}\left(\hat{T}^{a_1}\right) \leq \mathrm{ah}\left(\hat{T}^{A_2}\right) \leq \mathrm{ah}\left(\hat{T}^{a_3}\right).$$

By the assumption, a.a.s. as $n \to \infty$ we have $\mathrm{ah}(\hat{T}^{a_1}) \sim (1-\varepsilon)c \log(\ell_n)/2$ and $\mathrm{ah}(\hat{T}^{a_3}) \sim (1+\varepsilon)c \log(\ell_n)/2$. On the other hand, as noted above, T has the same distribution as $\hat{T}^{A_2}$. It follows that a.a.s. as $n \to \infty$

$$1 - 2\varepsilon \leq \frac{2\mathrm{ah}(T)}{c \log(\ell_n)} \leq 1 + 2\varepsilon.$$

Since ε was arbitrary, the result follows. ∎

It will be more convenient to view the process $\hat{P}$ in the following equivalent way. Denote by Exp (1) an exponential random variable with mean 1. Let $\hat{T}$ denote an infinite ternary tree whose nodes have types assigned using rules (a)–(d) and are associated with independent Exp (1) random variables. For convenience, each edge of the tree from a parent to a child is labelled with the random variable associated with the parent, which denotes the age of the parent when the child is born. For every node $u \in V(\hat{T})$, its *birth time* is defined as the sum of the labels on the edges connecting u to the root, and the birth time of the root is defined to be zero. Given $t \geq 0$, the tree $\hat{T}^t$ is the subtree induced by nodes with birth time less than or equal to t, and is finite with probability one.

Let $k \geq 3$ be a fixed positive integer. We define two random infinite trees $\underline{T_k}$ and $\overline{T_k}$ as follows. First, we regard $\hat{T}$ as a tree generated by each node giving birth to exactly three children with types assigned using (b)–(d), and with an Exp (1) random variable used to label the edges to its children. The tree $\underline{T_k}$ is obtained using the same generation rules as $\hat{T}$ except that every node of type 2 or 3, whose distance to its closest ancestor of type 1 is equal to k, dies without giving birth to any children. Given $t \geq 0$, the random (almost surely finite) tree $\underline{T_k^t}$ is, as before, the subtree of $\underline{T_k}$ induced by nodes with birth time less than or equal to t. The tree $\overline{T_k}$ is also generated similarly to $\hat{T}$, except that for each node u of type 2 (respectively, 3) in $\overline{T_k}$ whose distance to its closest ancestor of type 1 equals k, u has exactly three (respectively, four) children of type 1, and the edges joining u to its children get label 0 instead of random Exp (1) labels. (In the "evolving tree" interpretation, u immediately gives birth to three or four children of type 1 and dies.) Such a node u is called an *annoying* node. The random (almost surely finite) tree $\overline{T_k^t}$ is defined as before.

Lemma 5.3. *For every fixed $k \geq 3$, every $t \geq 0$, and every $g = g(t)$, we have*

$$\mathbb{P}\left[\mathrm{ah}\left(\underline{T_k^t}\right) \geq g\right] \leq \mathbb{P}\left[\mathrm{ah}\left(\hat{T}^t\right) \geq g\right] \leq \mathbb{P}\left[\mathrm{ah}\left(\overline{T_k^t}\right) \geq g\right].$$

Proof. The left inequality follows from the fact that the random edge labels of $\hat{T}$ and $\underline{T_k}$ can easily be coupled using a common sequence of independent Exp (1) random variables in such a way that for every $t \geq 0$, the generated $\underline{T_k^t}$ is always a subtree of the generated $\hat{T}^t$.

For the right inequality, we use a sneaky coupling between the edge labels of $\hat{T}$ and $\overline{T_k}$. It is enough to choose them using a common sequence of independent Exp (1) random variables $X_1, X_2, \ldots$ and define an injective mapping $f : V(\hat{T}) \to V\left(\overline{T_k}\right)$ such that for every $u \in V(\hat{T})$,

1. the auxiliary depth of $f(u)$ is greater than or equal to the auxiliary depth of u, and
2. for some I and $J \subseteq I$, the birth time of u equals $\sum_{i \in I} X_i$ and the birth time of $f(u)$ equals $\sum_{j \in J} X_j$.

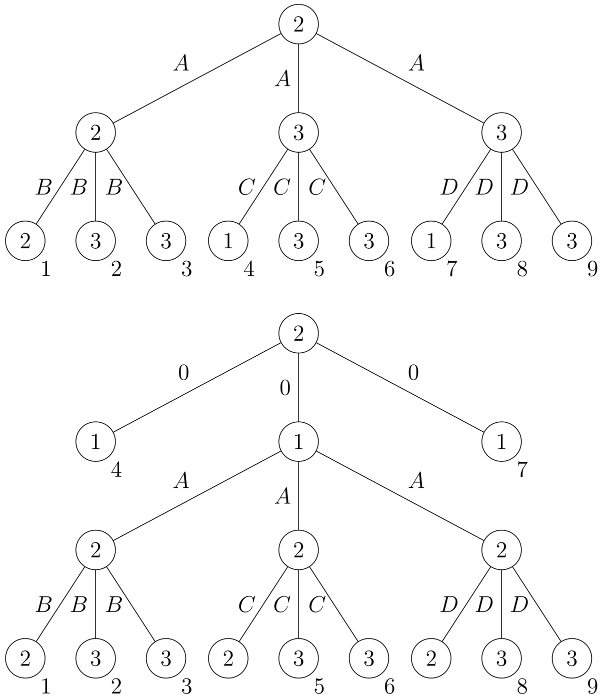


Fig. 4. Illustrating the coupling in Lemma 5.3 for an annoying node of type 2 in $\hat{T}$. The offspring of the node is shown above and the offspring of the corresponding node in $\overline{T_k}$ is shown below. The type of each node is written inside the node. The coupling of edge labels is defined by the appearance of $A, B, \dots$ in the two cases. The label 0 is also used in the case of $\overline{T_k}$. The function f is defined by the labels beside the nodes.

For annoying nodes, the coupling and the mapping f is shown down to their grandchildren in Figs. 4 and 5. This is easily extended in a natural way to all other nodes of the tree. ▪

With a view to proving Lemma 5.1 by appealing to Lemmas 5.2 and 5.3, we will define two sequences $(\underline{\rho_k})$ and $(\overline{\rho_k})$ such that for each k, a.a.s. the heights of $\overline{T_k^t}$ and $\underline{T_k^t}$ are asymptotic to $\overline{\rho_k}t$ and $\underline{\rho_k}t$, respectively, and also

$$\lim_{k\to\infty} \underline{\rho_k} = \lim_{k\to\infty} \overline{\rho_k} = c,$$

where $c \approx 1.668$ is defined in the statement of Lemma 5.1.

For the rest of this section, asymptotics are with respect to t instead of n, unless otherwise specified. We analyze the heights of $\underline{T_k}$ and $\overline{T_k}$ with the help of a theorem of Broutin and Devroye [10, Theorem 1]. We state here a special case suitable for our purposes, including a trivial correction to the conditions on E.

Theorem 5.4. *Let E be a template nonnegative random variable that satisfies $\mathbb{P}[E = 0] = 0$ and $\sup\{z : \mathbb{P}[E > z] = 1\} = 0$, and such that $\mathbb{P}[E = z] < 1$ for every $z \in \mathbb{R}$; and for which there exists $\lambda > 0$ such that $\mathbb{E}\left[\exp(\lambda E)\right]$ is finite. Let b be a fixed positive integer greater than 1 and let T_∞ be an infinite b-ary tree. Let B be a template random b-vector with each component distributed as E (but not necessarily independent components). For*

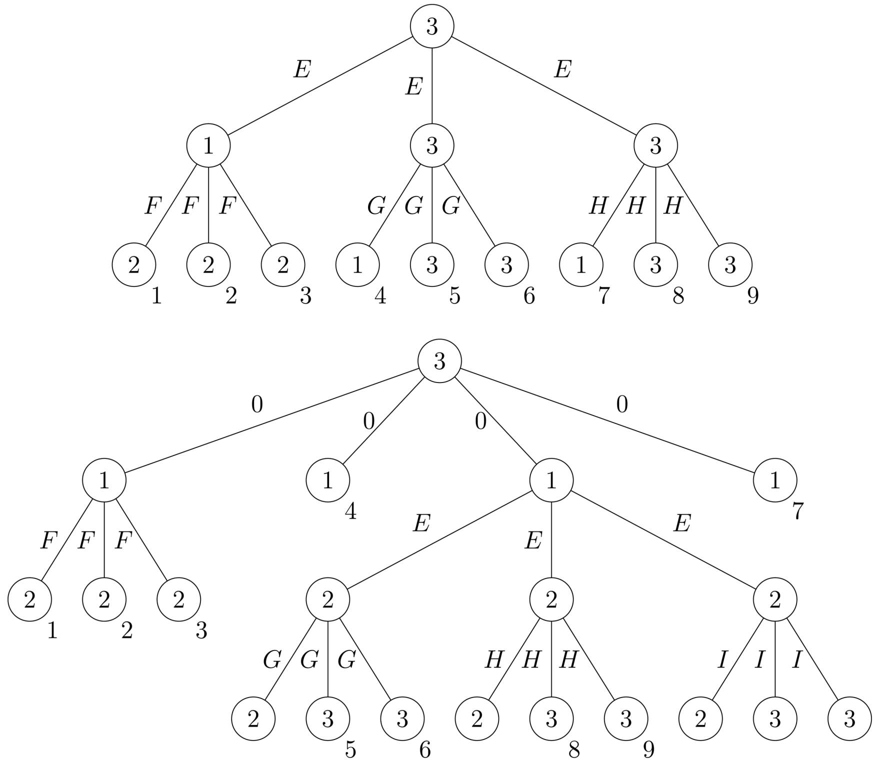


Fig. 5. The coupling in Lemma 5.3 for an annoying node of type 3 in $\hat{T}$.

every node u of T_∞, label the edges to the children of u using an independently generated copy of B.

Given $t \geq 0$, let H_t be the height of the subtree of T_∞ induced by the nodes for which the sum of the labels on their path to the root is at most t. Then, a.a.s. we have $H_t \sim \rho t$, where ρ is the unique solution to

$$\sup\{\lambda/\rho - \log\left(\mathbb{E}\left[\exp(\lambda E)\right]\right) : \lambda \leq 0\} = \log b.$$

For each $i = 2, 3, \ldots$, let $\alpha_i, \beta_i, \gamma_i$ denote the number of nodes of type 1, 2, 3 at depth i of $\hat{T}$ for which the root is the only node of type 1 in their path to the root. Then rules (a)–(d) for determining node types imply

$$\forall i > 2 \quad \alpha_i = \gamma_{i-1}, \quad \beta_i = \beta_{i-1}, \quad \gamma_i = 2\beta_{i-1} + 2\gamma_{i-1}.$$

These, together with $\alpha_2 = 0$, $\beta_2 = 3$, and $\gamma_2 = 6$, imply

$$\forall i \geq 2 \quad \alpha_i = 3 \times 2^{i-1} - 6, \quad \beta_i = 3, \quad \gamma_i = 3 \times 2^i - 6. \tag{5.1}$$

Let $\underline{b_k} = \sum_{i=1}^{k} \alpha_i$ and $\overline{b_k} = \sum_{i=1}^{k} \alpha_i + 3\beta_k + 4\gamma_k$. For a positive integer s, let Gamma (s) denote the Gamma distribution with mean s, i.e., the distribution of the sum of s independent Exp (1) random variables.

We define a random infinite tree $\underline{T_k}'$ as follows. The nodes of $\underline{T_k}'$ are the type-1 nodes of $\underline{T_k}$. Let V' denote the set of these nodes. For $u, v \in V'$ such that u is the closest type-1 ancestor of v in $\underline{T_k}$, there is an edge joining u and v in $\underline{T_k}'$, whose label equals the sum of the labels of the edges in the unique (u, v)-path in $\underline{T_k}$. By the construction, for all $t \geq 0$, the height of the subtree of $\underline{T_k}'$ induced by nodes with birth time less than or equal to t equals the auxiliary height of $\underline{T_k^t}$. Let u be a node in $\underline{T_k}'$. Then observe that for each $i = 3, 4, \ldots, k$, u has α_i children whose birth times equal the birth time of u plus a Gamma (i) random variable. In particular, $\underline{T_k}'$ is an infinite $\underline{b_k}$-ary tree.

To apply Theorem 5.4 we need the label of each edge to have the same distribution. For this, we create a random rearrangement of $\underline{T_k}'$. First let $\underline{E_k}$ be the random variable such that for each $3 \leq i \leq k$, with probability $\alpha_i/\underline{b_k}$, $\underline{E_k}$ is distributed as a Gamma (i) random variable. Now, for each node u of $\underline{T_k}'$, starting from the root and in a top-down manner, randomly permute the branches below u. This results in an infinite $\underline{b_k}$-ary tree, every edge of which has a random label distributed as $\underline{E_k}$. Although the labels of edges from a node to its children are dependent, the $\underline{b_k}$-vector of labels of edges from a node to its children is independent of all other edge labels, as required for Theorem 5.4. Let ρ be the solution to

$$\sup\{\lambda/\rho - \log\left(\mathbb{E}\left[\exp(\lambda \underline{E_k})\right]\right) : \lambda \leq 0\} = \log \underline{b_k}. \tag{5.2}$$

Then by Theorem 5.4, a.a.s. the auxiliary height of $\underline{T_k^t}$, which equals the height of the subtree of $\underline{T_k}'$ induced by nodes with birth time less than or equal to t, is asymptotic to ρt. Notice that we have

$$\mathbb{E}\left[\exp(\lambda \mathrm{Exp}\,(1))\right] = \frac{1}{1-\lambda}.$$

So, by the definition of Gamma (s), and since the product of expectation of independent variables equals the expectation of their product,

$$\mathbb{E}\left[\exp(\lambda \mathrm{Gamma}\,(s))\right] = \frac{1}{(1-\lambda)^s}.$$

Hence by linearity of expectation,

$$\mathbb{E}\left[\exp(\lambda \underline{E_k})\right] = \sum_{i=3}^{k} \frac{\alpha_i}{\underline{b_k}(1-\lambda)^i}. \tag{5.3}$$

One can define a random infinite $\overline{b_k}$-ary tree $\overline{T_k}'$ in a similar way. Let $\overline{E_k}$ be the random variable such that for each $3 \leq i \leq k-1$, with probability $\alpha_i/\overline{b_k}$, it is distributed as a Gamma (i) random variable, and with probability $(\alpha_k + 3\beta_k + 4\gamma_k)/\overline{b_k}$, it is distributed as a Gamma (k) random variable. Then by a similar argument, a.a.s. the auxiliary height of $\overline{T_k^t}$ is asymptotic to ρt, where ρ is the solution to

$$\sup\{\lambda/\rho - \log\left(\mathbb{E}\left[\exp(\lambda \overline{E_k})\right]\right) : \lambda \leq 0\} = \log \overline{b_k}. \tag{5.4}$$

Moreover, one calculates

$$\mathbb{E}\left[\exp(\lambda \overline{E_k})\right] = \frac{\alpha_k + 3\beta_k + 4\gamma_k}{\overline{b_k}(1-\lambda)^k} + \sum_{i=3}^{k-1} \frac{\alpha_i}{\overline{b_k}(1-\lambda)^i}. \tag{5.5}$$

As part of our plan to prove Lemma 5.1, we would like to define $\underline{\rho_k}$ and $\overline{\rho_k}$ in such a way that they are the unique solutions to (5.2) and (5.4), respectively. We first need to establish two analytical lemmas.

For later convenience, we define $\mathcal{F}$ to be the set of positive functions $f : [0.1, 0.2] \to \mathbb{R}$ that are differentiable on $(0.1, 0.2)$, and let $W : \mathcal{F} \to \mathbb{R}^{[0.1,0.2]}$ be the operator defined as

$$Wf(x) = x(x-1)f'(x)/f(x) - \log f(x).$$

Note that Wf is continuous. Define $h \in \mathcal{F}$ as

$$h(x) = \frac{12x^3}{1-2x} - \frac{6x^3}{1-x}.$$

Lemma 5.5. *The function Wh has a unique root $\hat{x}$ in (0.1,0.2).*

Proof. By the definition of $(\alpha_i)_{i\geq 3}$ in (5.1) we have

$$h(x) = \sum_{i\geq 3} \alpha_i x^i \quad \forall x \in [0.1, 0.2].$$

Since $\alpha_i > 0$ for all $i \geq 3$, we have $h(x) > 0$ and $h'(x) > 0$ for $x \in [0.1, 0.2]$, and hence the derivative of $\log h(x)$ is positive. Moreover, the derivative of $x(x-1)h'(x)/h(x)$ equals $4x(x-1)/(1-2x)^2$, which is negative. Therefore, $Wh(x)$ is a strictly decreasing function on [0.1,0.2]. Numerical calculations give $Wh(0.1) \approx 1.762 > 0$ and $Wh(0.2) \approx -0.831 < 0$. Hence, there is a unique solution to $Wh(x) = 0$ in (0.1,0.2). ∎

Remark. *Numerical calculations give $\hat{x} \approx 0.1629562$.*

Define functions $\underline{g_k}, \overline{g_k} \in \mathcal{F}$ as

$$\underline{g_k}(x) = \sum_{i=3}^{k} \alpha_i x^i, \quad \text{and} \quad \overline{g_k}(x) = (\alpha_k + 3\beta_k + 4\gamma_k)x^k + \sum_{i=3}^{k-1} \alpha_i x^i.$$

Note that by (5.3) and (5.5),

$$\underline{b_k}\, \mathbb{E}\left[\exp\left(\lambda \underline{E_k}\right)\right] = \underline{g_k}\left(\frac{1}{1-\lambda}\right), \quad \text{and} \quad \overline{b_k}\, \mathbb{E}\left[\exp\left(\lambda \overline{E_k}\right)\right] = \overline{g_k}\left(\frac{1}{1-\lambda}\right) \tag{5.6}$$

hold at least when $(1-\lambda)^{-1} \in [0.1, 0.2]$, namely for all $\lambda \in [-9, -4]$.

Lemma 5.6. *Both sequences $(W\underline{g_k})_{k=3}^{\infty}$ and $(W\overline{g_k})_{k=3}^{\infty}$ converge pointwise to Wh on $[0.1, 0.2]$ as $k \to \infty$. Also, there exists a positive integer k_0 and sequences $(\underline{x_k})_{k=k_0}^{\infty}$ and $(\overline{x_k})_{k=k_0}^{\infty}$ such that $W\underline{g_k}(\underline{x_k}) = W\overline{g_k}\,(\overline{x_k}) = 0$ for all $k \geq k_0$, and*

$$\lim_{k\to\infty} \underline{x_k} = \lim_{k\to\infty} \overline{x_k} = \hat{x}.$$

Proof. For any $x \in [0.1, 0.2]$, we have

$$\lim_{k\to\infty} \underline{g_k}(x) = h(x), \quad \lim_{k\to\infty} \underline{g_k}'(x) = h'(x), \quad \lim_{k\to\infty} \overline{g_k}(x) = h(x), \quad \lim_{k\to\infty} \overline{g_k}'(x) = h'(x),$$

so the sequences $(W\underline{g_k})_{k=3}^{\infty}$ and $(W\overline{g_k})_{k=3}^{\infty}$ converge pointwise to Wh.

Next, we show the existence of a positive integer $\underline{k_0}$ and a sequence $(\underline{x_k})_{k=\underline{k_0}}^{\infty}$ such that $W\underline{g_k}(\underline{x_k}) = 0$ for all $k \geq \underline{k_0}$, and

$$\lim_{k\to\infty} \underline{x_k} = \hat{x}.$$

The proof for existence of corresponding positive integer $\overline{k_0}$ and the sequence $(\overline{x_k})_{k=\overline{k_0}}^{\infty}$ is similar, and we may let $k_0 = \max\{\underline{k_0}, \overline{k_0}\}$.

Since $Wh(0.1) > 0$ and $Wh(0.2) < 0$, there exists $\underline{k_0}$ so that for $k \geq \underline{k_0}$, $W\underline{g_k}(0.1) > 0$ and $W\underline{g_k}(0.2) < 0$. Since $W\underline{g_k}$ is continuous for all $k \geq 3$, it has at least one root in $(0.1, 0.2)$. Moreover, since $W\underline{g_k}$ is continuous, the set $\{x : W\underline{g_k}(x) = 0\}$ is a closed set, thus we can choose a root $\underline{x_k}$ closest to $\hat{x}$. We just need to show that $\lim_{k\to\infty} \underline{x_k} = \hat{x}$. Fix an $\varepsilon > 0$. Since $Wh(\hat{x} - \varepsilon) > 0$ and $Wh(\hat{x} + \varepsilon) < 0$, there exists a large enough M such that for all $k \geq M$, $W\underline{g_k}(\hat{x} - \varepsilon) > 0$ and $W\underline{g_k}(\hat{x} + \varepsilon) < 0$. Thus $\underline{x_k} \in (\hat{x} - \varepsilon, \hat{x} + \varepsilon)$. Since ε was arbitrary, we conclude that $\lim_{k\to\infty} \underline{x_k} = \hat{x}$. ■

Let k_0 be as in Lemma 5.6 and let $(\underline{x_k})_{k=k_0}^{\infty}$ and $(\overline{x_k})_{k=k_0}^{\infty}$ be the sequences given by Lemma 5.6. Define the sequences $(\underline{\rho_k})_{k=k_0}^{\infty}$ and by

$$\underline{\rho_k} = \left(1 - \underline{x_k}^{-1}\right) / \log \underline{g_k}(\underline{x_k}), \quad \overline{\rho_k} = \left(1 - \overline{x_k}^{-1}\right) / \log \overline{g_k}(\overline{x_k}). \tag{5.7}$$

Lemma 5.7. *For every fixed $k \geq k_0$, a.a.s. the heights of $\overline{T_k^t}$ and $\underline{T_k^t}$ are asymptotic to $\overline{\rho_k}t$ and $\underline{\rho_k}t$, respectively.*

Proof. We give the argument for $\overline{T_k^t}$; the argument for $\underline{T_k^t}$ is similar. First of all, we claim that $\log(\mathbb{E}\left[\exp(\lambda\overline{E_k})\right])$ is a strictly convex function of λ over $(-\infty, 0]$. To see this, let $\lambda_1 < \lambda_2 \leq 0$ and let $\theta \in (0, 1)$. Then we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(\theta\lambda_1\overline{E_k} + (1-\theta)\lambda_2\overline{E_k}\right)\right] &= \mathbb{E}\left[\left[\exp\left(\lambda_1\overline{E_k}\right)\right]^{\theta}\left[\exp\left(\lambda_2\overline{E_k}\right)\right]^{1-\theta}\right] \\ &< \mathbb{E}\left[\exp\left(\lambda_1\overline{E_k}\right)\right]^{\theta}\mathbb{E}\left[\exp\left(\lambda_2\overline{E_k}\right)\right]^{1-\theta},\end{aligned}$$

where the inequality follows from Hölder's inequality, and is strict as the random variable $\overline{E_k}$ does not have all of its mass concentrated in a single point. Taking logarithms completes the proof of the claim.

It follows that given any value of ρ, $\lambda/\rho - \log(\mathbb{E}\left[\exp(\lambda\overline{E_k})\right])$ is a strictly concave function of $\lambda \in (-\infty, 0]$ and hence attains it supremum at a unique $\lambda \leq 0$.

Now, define

$$\overline{\lambda_k} = 1 - \overline{x_k}^{-1},$$

which is in $(-9, -4)$ as $\overline{x_k} \in (0.1, 0.2)$. Next we will show that

$$\overline{\lambda_k}/\overline{\rho_k} - \log\left(\mathbb{E}\left[\exp(\overline{\lambda_k}\overline{E_k})\right]\right) = \log \overline{b_k}, \tag{5.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda/\overline{\rho_k} - \log\left(\mathbb{E}\left[\exp(\lambda\overline{E_k})\right]\right)\right]\Big|_{\lambda=\overline{\lambda_k}} = 0, \tag{5.9}$$

which implies that $\overline{\rho_k}$ is the unique solution for (5.4), and thus by Theorem 5.4 and the discussion after it, the height of $\overline{T_k^t}$ is asymptotic to $\overline{\rho_k}t$.

Notice that $\overline{\lambda_k} \in (-9, -4)$, so by (5.6),

$$\overline{b_k}\mathbb{E}\left[\exp(\lambda\overline{E_k})\right] = \overline{g_k}((1-\lambda)^{-1})$$

for λ in a sufficiently small open neighbourhood of $\overline{\lambda_k}$. Taking logarithm of both sides and using (5.7) gives (5.8).

To prove (5.9), note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\log\left(\mathbb{E}\left[\exp(\lambda\overline{E_k})\right]\right)\right]\bigg|_{\lambda=\overline{\lambda_k}} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\log\,\overline{g_k}\left((1-\lambda)^{-1}\right) - \log\,\overline{b_k}\right]\bigg|_{\lambda=\overline{\lambda_k}} \\ &= \frac{\overline{g_k}'\left((1-\overline{\lambda_k})^{-1}\right)}{\left(1-\overline{\lambda_k}\right)^2\overline{g_k}\left((1-\overline{\lambda_k})^{-1}\right)} = \overline{x_k}^2\frac{\overline{g_k}'(\overline{x_k})}{\overline{g_k}(\overline{x_k})}.\end{aligned}$$

By Lemma 5.6, $W\overline{g_k}(\overline{x_k}) = 0$, i.e.,

$$\overline{x_k}^2\frac{\overline{g_k}'(\overline{x_k})}{\overline{g_k}(\overline{x_k})} = \overline{x_k}^2\frac{\log\,\overline{g_k}(\overline{x_k})}{\overline{x_k}(\overline{x_k}-1)} = \frac{\log\,\overline{g_k}(\overline{x_k})}{1-\overline{x_k}^{-1}} = \frac{1}{\overline{\rho_k}},$$

and (5.9) is proved. ∎

We now have all the ingredients to prove Lemma 5.1.

Proof of Lemma 5.1. By Lemma 5.2, we just need to show that a.a.s. the auxiliary height of $\hat{T}^t$ is asymptotic to ct, where

$$c = \frac{1-\hat{x}^{-1}}{\log\,h(\hat{x})}.$$

By Lemma 5.7, a.a.s. the heights of $\overline{T_k^t}$ and $\underline{T_k^t}$ are asymptotic to $\overline{\rho_k}t$ and $\underline{\rho_k}t$, respectively. By Lemma 5.6, $\overline{x_k} \to \hat{x}$ and $\underline{x_k} \to \hat{x}$. Observe that $(\underline{g_k})_{k=3}^{\infty}$ and $(\overline{g_k})_{k=3}^{\infty}$ converge pointwise to h, and that for every $k \geq 3$ and every $x \in [0.1, 0.2]$, $\underline{g_k}(x) \leq \underline{g_{k+1}}(x)$ and $\overline{g_k}(x) \geq \overline{g_{k+1}}(x)$. Thus by Dini's theorem (see, e.g., Rudin [16, Theorem 7.13]), $(\underline{g_k})_{k=3}^{\infty}$ and $(\overline{g_k})_{k=3}^{\infty}$ converge uniformly to h on $[0.1, 0.2]$.

Hence,

$$\lim_{k\to\infty}\underline{\rho_k} = \lim_{k\to\infty}\frac{1-\underline{x_k}^{-1}}{\log\,\underline{g_k}(\underline{x_k})} = \frac{1-\hat{x}^{-1}}{\log\,h(\hat{x})} = c,$$

and

$$\lim_{k\to\infty}\overline{\rho_k} = \lim_{k\to\infty}\frac{1-\overline{x_k}^{-1}}{\log\,\overline{g_k}(\overline{x_k})} = \frac{1-\hat{x}^{-1}}{\log\,h(\hat{x})} = c.$$

It follows from Lemma 5.3 that a.a.s. the auxiliary height of $\hat{T}^t$ is asymptotic to ct, as required. ∎

Notes added in proof. Using a rather involved argument, Chen and Yu [17, Corollary 3.5] have shown that every 3-connected planar graph on n vertices has a cycle with at least

$n^{\log 2/\log 3}$ vertices. By Theorem 1.2(a), every n-vertex random apollonian network has a cycle with at least $(2n-5)^{\log 2/\log 3}+2$ vertices deterministically, which gives a slightly better result for this subclass, having the same exponent but a larger constant. The example studied in [17, Section 2] shows that the exponent $\log 2/\log 3$ here is the best possible for a deterministic lower bound. We thank David Wood for pointing out the paper [17] to us.

Very recently, Cooper and Frieze announced that they have independently proved using different techniques that a.a.s. the diameter of a RAN on n vertices is asymptotic to $c' \log n$, where c' satisfies a certain implicit equation. The value of c' seems to equal the value of c in Theorem 1.3 numerically, but there is no obvious direct way to see this.

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