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Electronic Notes in Discrete Mathematics 43 (2013) 355–365

Electronic Notes in DISCRETE MATHEMATICS

www.elsevier.com/locate/endm

# On the Longest Paths and the Diameter in Random Apollonian Networks [1]

Ehsan Ebrahimzadeh [a,2], Linda Farczadi [b,3], Pu Gao [c,4], Abbas Mehrabian [b,5], Cristiane M. Sato [b,6], Nick Wormald [b,d,7,8] and Jonathan Zung [e,9]

[a] *Department of Electrical and Computer Engineering, University of Waterloo, Waterloo ON, Canada*

[b] *Department of Combinatorics and Optimization, University of Waterloo, Waterloo ON, Canada*

[c] *Department of Computer Science, University of Toronto, Toronto ON, Canada*

[d] *School of Mathematical Sciences, Monash University, Clayton VIC, Australia*

[e] *Department of Mathematics, University of Toronto, Toronto ON, Canada*

**Abstract**

We consider the following iterative construction of a random planar triangulation. Start with a triangle embedded in the plane. In each step, choose a bounded face uniformly at random, add a vertex inside that face and join it to the vertices of the face. After $n-3$ steps, we obtain a random triangulated plane graph with $n$ vertices, which is called a Random Apollonian Network (RAN). We show that asymptotically almost surely (a.a.s.) every path in a RAN has length $o(n)$, refuting a conjecture of Frieze and Tsourakakis. We also show that a RAN always has a path of length $(2n-5)^{\log 2/\log 3}$, and that the expected length of its longest path is $\Omega\left(n^{0.88}\right)$. Finally, we prove that a.a.s. the diameter of a RAN is asymptotic to $c\log n$, where $c \approx 1.668$ is the solution of an explicit equation.

*Keywords:* random apollonian networks, random plane graphs, longest paths, heights of random trees


http://dx.doi.org/10.1016/j.endm.2013.07.053

# 1 Introduction

Due to the ever growing interest in social networks, the Web graph, biological networks, etc., in recent years a great deal of research has been built around modelling real world networks (see, e.g., the surveys [5,8,15] or the books [4,6,10]). Despite the outstanding amount of work on models generating graphs with power law degree sequences, a considerably smaller amount of work has focused on generative models for planar graphs. In this paper we study a popular random graph model for generating planar graphs with power law properties, which is defined as follows. Start with a triangle embedded in the plane. In each step, choose a bounded face uniformly at random, add a vertex inside that face and join it to the vertices of the face. We call this operation *subdividing* a face. Throughout the paper, we use the term "face" to refer to a "bounded face," unless specified otherwise. After $n-3$ steps, we have a (random) triangulated plane graph with $n$ vertices and $2n-5$ faces. This is called a *Random Apollonian Network (RAN)* and we study its asymptotic properties as its number of vertices goes to infinity. The number of edges equals $3n-6$; hence a RAN is a maximal plane graph.

The term "apollonian network" refers to a deterministic version of this process, formed by subdividing all the triangles in the same level the same number of times, which was first studied in [2,11]. Andrade et al. [2] studied power laws in the degree sequences of these networks. Random apollonian networks were defined in [19] (see also [18] for a generalization to higher dimensions), where it was proved that the diameter of a RAN is probabilistically bounded above by a constant times the logarithm of the number of vertices. It was shown in [19,17] that RANs exhibit a power law degree distribution.

[1] A detailed version of this paper containing all the proofs is available in http://arxiv.org/abs/1303.5213

[2] Email: `eebrahim@uwaterloo.ca`
[3] Email: `lindafarczadi@gmail.com`
[4] Email: `jane.p.gao@gmail.com`
[5] Email: `amehrabi@uwaterloo.ca`
[6] Email: `cmsato@gmail.com`
[7] Email: `nick.wormald@monash.edu`

[8] This author acknowledges support from the Canada Research Chairs program and NSERC, and the ARC Australian Laureate Fellowship scheme.

[9] Email: `jonathanzung@gmail.com`

The average distance between two vertices in a typical RAN was shown to be logarithmic [1]. The degree distribution, $k$ largest degrees and $k$ largest eigenvalues (for fixed $k$) and the diameter were studied in [14]. We continue this line of research by studying the asymptotic properties of the longest (simple) paths in RANs and giving sharp estimates for the diameter of a typical RAN.

Before stating our main results, we need a few definitions. In this paper $n$ and $m$ denote the number of vertices and faces of the RAN, respectively. All logarithms are in the natural base. We say an event $A$ happens *asymptotically almost surely (a.a.s.)* if $\mathbb{P}[A]$ approaches 1 as $n$ goes to infinity. For a random variable $X = X(n)$ and a function $f(n)$, we say *a.a.s.* $X = o\big(f(n)\big)$ if for every fixed $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}[X \le \varepsilon f(n)] = 1$ . Similarly, we say $X$ is a.a.s. asymptotic to $f$ if a.a.s. $X = \big(1 + o(1)\big) f$.

In the concluding remarks of [14], Frieze and Tsourakakis conjectured that a.a.s. a RAN has a path of length $\Omega(n)$. We refute this conjecture by proving the following theorem. Let $\mathcal{L}_m$ be a random variable denoting the number of vertices in a longest path of a RAN with $m$ faces.

**Theorem 1.1** *A.a.s. we have $\mathcal{L}_m = o(m)$.*

Recall that a RAN on $n$ vertices has $2n - 5$ faces, so Theorem 1.1 implies that a.a.s. a RAN does not have a path of length $\Omega(n)$.

We also prove a deterministic lower bound for the length of a longest path, as well as a lower bound for its expected value.

**Theorem 1.2** *For every positive integer $m$, the following statements are true.*

*(a)*

$$\mathcal{L}_m \ge m^{\log 2/\log 3} + 2 \,.$$

*(b)*

$$\mathbb{E}\left[\mathcal{L}_m\right] = \Omega\left(m^{0.88}\right) \,.$$

The proofs of Theorems 1.1 and 1.2 are built on two novel graph theoretic observations, valid for all subgraphs of an apollonian network.

We also study the diameter of RANs. Frieze and Tsourakakis [14] showed that the diameter of a RAN is a.a.s. at most $\eta_2 \log n$, where $\eta_2 \approx 7.081$ is the unique solution greater than 1 of $\exp(1/x) = 3e/x$. (Our statement here corrects a minor error in [14], propagated from Broutin and Devroye [7], which stated that $\eta_2$ is the unique solution less than 1.) Albenque and Marckert [1] showed that a.a.s. the distance between two randomly chosen vertices of a RAN (which naturally gives a lower bound on the diameter) is asymptotic to $\eta_1 \log n$, where $\eta_1 = 6/11 \approx 0.545$. In this paper, we provide the asymptotic value for the diameter of a typical RAN.

**Theorem 1.3** *A.a.s. the diameter of a RAN on $n$ vertices is asymptotic to $c\log n$, with $c = (1-\hat{x}^{-1})/\log h(\hat{x}) \approx 1.668$, where*

$$h(x) = \frac{12x^3}{1-2x} - \frac{6x^3}{1-x}\,,$$

*and $\hat{x} \approx 0.163$ is the unique solution in the interval $(0.1, 0.2)$ to $x(x-1)h'(x) = h(x)\log h(x)$ .*

The proof of Theorem 1.3 consists of a nontrivial reduction of the problem of estimating the diameter to the problem of estimating the height of a certain skewed random tree, which can be done by applying a result of Broutin and Devroye [7].

We include some definitions here. Let $G$ be a RAN. We denote the vertices incident with the unbounded face by $\nu_1, \nu_2, \nu_3$. We define a rooted tree $T$, called the $\triangle$-*tree* of $G$, as follows. There is a one to one correspondence between the triangles in $G$ and the nodes of $T$. For every triangle $\triangle$ in $G$, we denote its corresponding node in $T$ by $\mathbf{n}^{\triangle}$. To build $T$, start with a single root node, which corresponds to the triangle $\nu_1\nu_2\nu_3$ of $G$. Wherever a triangle $\triangle$ is subdivided into triangles $\triangle_1$, $\triangle_2$, and $\triangle_3$, generate three children $\mathbf{n}^{\triangle_1}$, $\mathbf{n}^{\triangle_2}$, and $\mathbf{n}^{\triangle_3}$ for $\mathbf{n}^{\triangle}$, and extend the correspondence in the natural manner. Note that this is a random ternary tree, with each node having either zero or three children, and has $3n-8$ nodes and $2n-5$ leaves. We use the term "nodes" for the vertices of $T$, so that "vertices" refer to the vertices of $G$. The *depth* of a node $\mathbf{n}^{\triangle}$ is its distance to the root. Note that the leaves of $T$ correspond to the faces of $G$.

We sketch the proofs Theorems 1.1, 1.2, and 1.3 in Sections 2, 3, and 4, respectively.

## 2 Upper bound for the longest paths

Let $\triangle$ be a triangle in a RAN. The *1-subdivision* of $\triangle$ is the set $\{\triangle_1, \triangle_2, \triangle_3\}$ of three triangles obtained from subdividing $\triangle$ once, and the 2-*subdivision* of $\triangle$ is obtained from subdividing each of these three triangles exactly once. We can analyze the number of faces inside $\triangle_1$ by modelling the process of building the RAN as an *Eggenberger-Pólya urn* (see, e.g., [16, Section 5.1]) in a natural way. Then a result of Eggenberger and Pólya [13] (see also [16, Theorem 5.1.2]) implies that the probability distribution function of the proportion of faces in

$\triangle_1$ converges pointwise to that of a beta random variable with parameters 1/2 and 1. In this way we obtain the following.

**Lemma 2.1** *Let $\triangle$ be a triangle containing $m$ faces in a RAN, and let $Z_1, Z_2, \ldots, Z_9$ be the number of faces inside the nine triangles in the 2-subdivision of $\triangle$. Given $\varepsilon > 0$, there exists $m_0 = m_0(\varepsilon)$ such that for $m > m_0$,*

$$\mathbb{P}\left[\min\{Z_1, \ldots, Z_9\}/m < \varepsilon\right] < 13\sqrt[4]{\varepsilon}\,.$$

The set of *grandchildren* of a node is the set of children of its children, so every node in a ternary tree has between zero and nine grandchildren. The following lemma can be proved easily.

**Lemma 2.2** *Let $G$ be a RAN and let $T$ be its $\triangle$-tree. Let $\mathbf{n}^{\triangle}$ be a node of $T$ with nine grandchildren $\mathbf{n}^{\triangle_1}, \mathbf{n}^{\triangle_2}, \ldots, \mathbf{n}^{\triangle_9}$. Then a path in $G$ cannot enter the interior of each triangle $\triangle_1, \ldots, \triangle_9$.*

**Proof of Theorem 1.1 (sketch).** Let $\triangle$ be the triangle $\nu_1\nu_2\nu_3$. The 2-subdivision of $\triangle$ consists of nine triangles, and every path misses the vertices in at least one of them by Lemma 2.2. We can now apply the same argument inductively for the other eight triangles, and repeat. Note that if the distribution of vertices in the nine triangles of every 2-subdivision were always moderately balanced, this argument would immediately prove the theorem (by repeating $O(\log n)$ steps). Unfortunately, the distribution is biased towards becoming unbalanced: the greater the number of vertices falling in a certain triangle, the higher the probability that the next vertex falls in the same triangle.

However, Lemma 2.1 gives an upper bound for the probability that this distribution is very unbalanced. Let $k = (\log\log n)/2$. Let $d_0 = 0$ and $d_i = 2^{i-1}k$ for $1 \le i \le k$. It can be proved using Chauvin and Drmota [9, Theorem 2.3] (see also Drmota [12, Theorem 6.47]), that the tree $T$ is a.a.s. full down to level $2d_k$. Let $\varepsilon$ be a fixed positive number such that $3(13\sqrt[4]{4\varepsilon})^{1/5} < 1$. Using Lemma 2.1 one can prove the following holds a.a.s.

Let $v$ be an arbitrary node of $T$ at depth $d_i$ for some $1 \le i \le k$, and let $u$ be the ancestor of $v$ at depth $d_{i-1}$. Then there is at least one node $f$ on the $(u, v)$-path in $T$ with depth between $d_{i-1}$ and $d_i - 2$, such that $f$ has nine grandchildren, each of whose triangles contains at least an $\varepsilon$ fraction of the vertices in $f$'s triangle.

The rest of the proof is straightforward by applying this result iteratively for $i = 1, 2, \ldots, k$, and keeping track of the number vertices left out from the path in each iteration. □

# 3 Lower bounds for the longest paths

Let $G$ be a RAN with $m$ faces, and let $v$ be the unique vertex that is adjacent to $\nu_1$, $\nu_2$, and $\nu_3$. For $1 \leq i \leq 3$, let $\triangle_i$ be the triangle with vertex set $\{v, \nu_1, \nu_2, \nu_3\} \setminus \{\nu_i\}$. Define the random variable $\mathcal{L}'_m$ as the largest number $L$ such that for every permutation $\pi$ on $\{1, 2, 3\}$, there is a path in $G$ of $L$ edges from $\nu_{\pi(1)}$ to $\nu_{\pi(2)}$ not containing $\nu_{\pi(3)}$. Clearly we have $\mathcal{L}_m \geq \mathcal{L}'_m + 2$.

**Proof of Theorem 1.2(a).** Let $\xi = \log 2/\log 3$. We prove by induction on $m$ that $\mathcal{L}'_m \geq m^{\xi}$. The induction base is obvious for $m = 1$, so assume that $m > 1$. Let $m_i$ denote the number of faces in $\triangle_i$. Then $m_1+m_2+m_3 = m$. By symmetry, we may assume that $m_1 \geq m_2 \geq m_3$. For any given $1 \leq i \leq 3$, it is easy to find a path avoiding $\nu_i$ that connects the other two $\nu_j$'s by attaching two appropriate paths in $\triangle_1$ and $\triangle_2$ at vertex $v$. By the inductive hypothesis, these paths can be chosen to have lengths at least ${m_1}^{\xi}$ and ${m_2}^{\xi}$. Hence for every permutation $\pi$ on $\{1, 2, 3\}$, there is a path from $\nu_{\pi(1)}$ to $\nu_{\pi(2)}$ avoiding $\nu_{\pi(3)}$ with length at least ${m_1}^{\xi} + {m_2}^{\xi}$. This quantity is minimized when $m_1 = m_2 = m/3$, thus $\mathcal{L}'_m \geq {m_1}^{\xi} + {m_2}^{\xi} \geq 2\,(m/3)^{\xi} = m^{\xi}$. □

**Proof of Theorem 1.2(b) (sketch).** Let $\zeta = 0.88$. We prove by induction on $m$ that for some fixed $\kappa$ we have $\mathbb{E}\left[\mathcal{L}'_m\right] \geq \kappa m^{\zeta}$. By choosing $\kappa$ small enough, we may assume that this holds for all $m$ smaller than a certain number. Let the random variable $X_i$ denote the number of faces in $\triangle_i$. Define a random permutation $\sigma$ on $\{1, 2, 3\}$ such that $X_{\sigma(1)} \geq X_{\sigma(2)} \geq X_{\sigma(3)}$, breaking ties randomly. Then we have

$$\begin{aligned}\mathbb{E}\left[\mathcal{L}'_m\right] \geq \mathbb{E}\left[\mathcal{L}'_{X_{\sigma(1)}} + \mathcal{L}'_{X_{\sigma(2)}}\right] &\geq 6\mathbb{E}\left[(\mathcal{L}'_{X_1} + \mathcal{L}'_{X_2})\mathbb{1}_{X_1>X_2>X_3}\right] \\ &\geq 6\kappa\mathbb{E}\left[(X_1^{\zeta} + X_2^{\zeta})\mathbb{1}_{X_1>X_2>X_3}\right],\end{aligned}$$

using first an argument similar to that of part (a), then symmetry, and then the inductive hypothesis.

By the distributional result of Eggenberger and Pólya [13] (mentioned in Section 2), the distribution of $\frac{X_i}{m}$ converges pointwise to that of a beta random variable with parameters $\frac{1}{2}$ and 1. Moreover, for any fixed $\varepsilon \in [0, 1)$, the distribution of $\frac{X_2}{(1-\varepsilon)m}$ conditional on $X_1 = \varepsilon m$ converges pointwise to that of a beta random variable with parameters $\frac{1}{2}$ and $\frac{1}{2}$ . Let $f_1(x)$ (respectively, $f_2(x)$) denote the probability density function of a beta random variable with parameters $\frac{1}{2}$ and 1 (respectively, $\frac{1}{2}$ and $\frac{1}{2}$). Hence (see Billingsley [3, Theorem 29.1(i)])

$$\mathbb{E}\left[\left(\left(\frac{X_1}{m}\right)^{\zeta}+\left(\frac{X_2}{m}\right)^{\zeta}\right)\mathbb{1}_{X_1>X_2>X_3}\right]$$
$$\to \int_{t=1/3}^{1}\int_{s=1/2}^{\min\{1,\frac{t}{1-t}\}}\left[t^{\zeta}+(s(1-t))^{\zeta}\right]f_1(t)f_2(s)\,\mathrm{d}s\mathrm{d}t > 1/6\,,$$

as required. □

# 4 Diameter

Let $G$ be a RAN with $n$ vertices. For a vertex $v$ of $G$, let $\tau(v)$ be the minimum graph distance between $v$ and $\nu_1$, $\nu_2$ or $\nu_3$. The *radius* of $G$ is defined as the maximum of $\tau(v)$ over all vertices $v$. We will sketch the proof that the radius of $G$ is a.a.s. asymptotic to $\frac{c}{2}\log n$; it is not hard to show that Theorem 1.3 follows from this. Let $T$ be the $\triangle$-tree of $G$. Any triangle $\triangle$ in $G$ with vertex set $\{x, y, z\}$ such that $\tau(x) \le \tau(y) \le \tau(z)$ can be categorized as type 1 if $\tau(x) = \tau(y) = \tau(z)$; type 2 if $\tau(x) = \tau(y) < \tau(y) + 1 = \tau(z)$; and type 3 if $\tau(x) < \tau(x) + 1 = \tau(y) = \tau(z)$. The type of a node of $T$ is defined to be the same as the type of its corresponding triangle. The following are easy to observe.

(a) The root is of type 1.

(b) A node of type 1 has three children of type 2.

(c) A node of type 2 has one child of type 2 and two children of type 3.

(d) A node of type 3 has two children of type 3 and one child of type 1.

For a triangle $\triangle$, define $\tau(\triangle)$ to be the minimum of $\tau(u)$ for all $u \in V(\triangle)$. Then it is not hard to observe that for every $\mathbf{n}^{\triangle} \in V(T)$, $\tau(\triangle)$ is one less than the number of nodes of type 1 in the path from $\mathbf{n}^{\triangle}$ to the root of $T$. We call the quantity $\tau(\triangle)$ the *auxiliary depth* of $\mathbf{n}^{\triangle}$, and define the *auxiliary height* of a tree $T$, written $\mathrm{ah}(T)$, to be the maximum auxiliary depth of its nodes. Hence, the radius of a RAN equals one plus the auxiliary height of its $\triangle$-tree.

Notice that instead of building $T$ from the RAN $G$, one can think of the random $T$ as being generated in the following manner: start with a single node as the root of $T$. So long as the number of nodes is less than $3n-8$, choose a leaf $v$ independently of previous choices and uniformly at random, and add three leaves as children of $v$. Once the number of nodes becomes $3n-8$, assign types to the nodes subject to rules (a)–(d), using independent random choices to decide which of the children have the particular types. Henceforth, we will forget about $G$ and focus on finding the auxiliary height of a random tree $T$ generated in this manner.

A major difficulty in analyzing the auxiliary height of such a tree $T$ is that the subtrees rooted at the children of a node are heavily dependent, as the total number of nodes is $3n-8$. To remedy this we consider another random tree which has the desired independence and approximates $T$ well enough for our purposes. Denote by Exp(1) an exponential random variable with mean 1. Let $\widehat{T}$ denote an infinite ternary tree whose nodes have types assigned using rules (a)–(d) and are associated with independent Exp(1) random variables. For the sake of convenience, each edge of the tree from a parent to a child is labelled with the random variable associated with the parent, which can be viewed as the age of the parent when the child is born. For every node $u \in V(\widehat{T})$, its *birth time* is defined as the sum of the labels on the edges connecting $u$ to the root, and the birth time of the root is defined to be zero. Given $t \geq 0$, the tree $\widehat{T}^t$ is the subtree induced by nodes whose birth times are less than or equal to $t$, and is finite with probability one. By the memorylessness of the exponential distribution, for any deterministic $t \geq 0$, the distribution of $\widehat{T}^t$ conditional on $\widehat{T}^t$ having exactly $3n-8$ nodes, is the same as the distribution of $T$.

Using Broutin and Devroye [7, Proposition 2], it is easy to deduce the desired result on the radius of a RAN after proving that a.a.s. the auxiliary height of $\widehat{T}^t$ is asymptotic to $ct$ as $t \to \infty$.

Let $k \geq 3$ be a fixed positive integer. We define two random infinite trees $\underline{T_k}$ and $\overline{T_k}$ as follows. First, we regard $\widehat{T}$ as a tree generated by each node giving birth to exactly three children with types assigned using (a)–(d), and with an Exp(1) random variable used to label the edges to its children. The tree $\underline{T_k}$ is obtained using the same generation rules as $\widehat{T}$ except that every node of type 2 or 3, whose distance to its closest ancestor of type 1 is equal to $k$, gives birth to no children. Given $t \geq 0$, the (finite) tree $\underline{T_k^t}$ is, as before, the subtree of $\underline{T_k}$ induced by nodes whose birth times are less than or equal to $t$. The tree $\overline{T_k}$ is also generated similarly to $\widehat{T}$, except that for each node $u$ of type 2 (respectively, 3) in $\overline{T_k}$ whose distance to its closest ancestor of type 1 equals $k$, $u$ has exactly three (respectively, four) children of type 1, and the edges joining $u$ to its children get label 0 instead of random Exp(1) labels. (In an "evolving tree" interpretation, $u$ immediately gives birth to three or four children of type 1 and dies.) The (finite) tree $\overline{T_k^t}$ is defined as before. The following "sandwiching" lemma implies that we just need to analyze the trees

$\underline{T_k^t}$ and $\overline{T_k^t}$.

**Lemma 4.1** *For every positive constant $k \geq 3$, every $t \geq 0$, and every $g = g(t)$, we have*

$$\mathbb{P}\left[\mathrm{ah}\left(\underline{T_k^t}\right) \geq g\right] \leq \mathbb{P}\left[\mathrm{ah}\left(\widehat{T}^t\right) \geq g\right] \leq \mathbb{P}\left[\mathrm{ah}\left(\overline{T_k^t}\right) \geq g\right] .$$

**Proof Sketch.** The left inequality follows from the fact that the random edge labels of $\widehat{T}$ and $\underline{T_k}$ can easily be coupled using a common sequence of independent Exp(1) random variables in such a way that for every $t \geq 0$, the generated $\underline{T_k^t}$ is always a subtree of the generated $\widehat{T}^t$. The right inequality is proved by defining a sneaky coupling between the edge labels of $\widehat{T}$ and $\overline{T_k}$, which is omitted from this abstract. □

**Proof of Theorem 1.3 (sketch).** Asymptotics are with respect to $t$ instead of $n$. We define a random infinite tree $\underline{T_k}'$ as follows. The nodes of $\underline{T_k}'$ are the type-1 nodes of $\underline{T_k}$. Let $V'$ denote the set of these nodes. For $u, v \in V'$ such that $u$ is the closest type-1 ancestor of $v$ in $\underline{T_k}$, we have an edge joining $u$ and $v$ in $\underline{T_k}'$, whose label equals the sum of the labels of the edges in the unique $(u, v)$-path in $\underline{T_k}$.

To apply [7, Theorem 1] we need the label of each edge to have the same distribution. For this, we create a certain random rearrangement of $\underline{T_k}'$. In this new tree, although the labels of edges from a node to its children are dependent, the vector of labels of edges from a node to its children is independent of all other edge labels, as required for [7, Theorem 1]. By this theorem, a.a.s. the height of the subtree of $\underline{T_k}'$ induced by nodes whose birth times are less than or equal to $t$ is asymptotic to $\underline{\rho_k} t$ for a certain constant $\underline{\rho_k}$. By the construction of $\underline{T_k}'$, this height equals the auxiliary height of $\underline{T_k^t}$.

One can define an infinite $\overline{b_k}$-ary tree $\overline{T_k}'$ in a similar way. It follows by a similar argument that a.a.s. the auxiliary height of $\overline{T_k^t}$ is asymptotic to $\overline{\rho_k} t$, for a certain constant $\overline{\rho_k}$. With some analysis, we can show that $\lim_{k\to\infty} \underline{\rho_k} = \lim_{k\to\infty} \overline{\rho_k} = c$. It then follows from Lemma 4.1 that a.a.s. the auxiliary height of $\widehat{T}^t$ is asymptotic to $ct$, as required. □

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